Ma 416: Complex Variables
Homework Assignment 12

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Due Thursday, December 8th, 2005

Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 4, sections 21A–23B and 25A-25E.

1. Suppose that $u = u(x, y)$ is continuous on the closed unit disk $D = \{(x, y) : x^2+y^2 \leq 1\}$ and u is twice continuously differentiable with $\Delta u(x, y) = 0$ inside D. Find a series representing u for each of the following boundary conditions $u(\cos\theta, \sin\theta) = \psi(\theta)$, $-\pi \leq \theta \leq \pi$:

$$(a) \quad \psi(\theta) = \begin{cases} 1, & \text{if } \theta \in [-\pi/2, \pi/2]; \\ 0, & \text{otherwise,} \end{cases} \qquad\qquad (b) \quad \psi(\theta) = \sin\theta.$$

2. Suppose that $u(x, y)$ is continuous on the closed annulus $A = \{(x, y) : 1 \leq x^2+y^2 \leq 4\}$ and u is twice continuously differentiable with $\Delta u(x, y) = 0$ inside A. Find a series representing u for each of the following boundary conditions $u(\cos\theta, \sin\theta) = \psi_1(\theta)$ and $u(2\cos\theta, 2\sin\theta) = \psi_2(\theta)$, $-\pi \leq \theta \leq \pi$:

$$(a) \quad \psi_1(\theta) = 0;\ \psi_2(\theta) = |\theta|; \qquad\qquad (b) \quad \psi_1(\theta) = \cos\theta;\ \psi_2(\theta) = \sin\theta$$

3. Suppose that $f = f(z)$ is analytic and univalent in a region $D \subset \mathbf{C}$ and let $E = f(D) = \{f(z) : z \in D\}$ be its range. Write $u + iv = f(x + iy)$ and identify (u, v) with $u + iv$. Prove that if $\phi = \phi(u, v)$ is a harmonic function in E, then $\psi(x, y) \stackrel{\text{def}}{=} \phi(f(x + iy))$ is a harmonic function in D.

4. Find a Möbius transform mapping $0, 1, i$ to $\infty, 1, -i$, respectively. Is it unique?

5. Find all the Möbius transforms that the unit disk $\{|z| < 1\}$ to its exterior $\{|z| > 1\}$.