Ma 416: Complex Variables
Solutions to Homework Assignment 2

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Due Thursday, September 15th, 2005

1. Prove or find a counterexample to the following statements:

   (a) If $f(x) = O(g(x))$ as $x \to 0$, then $f(x)/g(x) \to 0$ as $x \to 0$.

   (b) If $f(x) = o(g(x))$ as $x \to \infty$, then $f(x)/[1 + |g(x)|] \to 0$ as $x \to \infty$.

   (c) If $f(x) = o(g(x))$ as $x \to 1$, then $f(x) = O(g(x))$ as $x \to 1$.

   (d) If $f(x) = o(x)$ as $x \to 0$, then $f(x) = O(x^2)$ as $x \to 0$.

   Solution: (a) This is false. Let $f(x) = g(x) = 1$ for all x; then $f(x)/g(x) = 1$ for all x and cannot have 0 as a limit as $x \to 0$.

   (b) This is true. By definition, $f(x) = o(g(x))$ as $x \to \infty$ implies that $f(x)/g(x) \to 0$ as $x \to \infty$. But then $|f(x)/g(x)| \to 0$ as $x \to \infty$, and

   $$0 \leq \frac{|f(x)|}{1 + |g(x)|} \leq \frac{|f(x)|}{|g(x)|} \to 0, \qquad \text{as } x \to \infty,$$

   so the result follows from the squeeze law of limits.

   (c) This is true. The hypothesis $f(x) = o(g(x))$ as $x \to \infty$ implies that $|f(x)/g(x) \to 0$ as $x \to \infty$. But then for any $\epsilon > 0$ there must be some $\delta > 0$ such that $|f(x)/g(x)| \leq \epsilon$ for all x satisfying $|x - 1| < \delta$. Choosing $\epsilon = 2$ and finding the corresponding δ yields the result:

   $$|f(x)| \leq 2|g(x)| \qquad \text{for all } x \text{ with } |x - 1| < \delta,$$

   which is a particular case of the statement $f(x) = O(g(x))$ as $x \to 1$.

   (d) This is false. The function $f(x) = x\sqrt{|x|}$ satisfies $f(x) = o(x)$ as $x \to 0$ but not $f(x) = O(x^2)$ as $x \to 0$, since $|f(x)/x^2| = 1/\sqrt{|x|}$ is not bounded in any neighborhood of $x = 0$. □

2. Let $f(x, y) = u(x, y) + iv(x, y)$ be a complex-valued function of two real variables. Write $z = x + iy$ for the complex variable with real part x and imaginary part y. Show that the Cauchy-Riemann equations are equivalent to the equation

   $$\frac{\partial}{\partial \bar{z}} f(z) = 0,$$

   using the defintion $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]$ on page 19 of our textbook.

   Solution: Compute

   $$\frac{\partial}{\partial \bar{z}} f(z) = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right]$$

   If $\frac{\partial}{\partial \bar{z}} f(z) = 0$, then both the real and imaginary parts of the derivative must be zero, so

   $$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0; \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

These are the Cauchy-Riemann equations.

Conversely, if the Cauchy-Riemann equations are satisfied by u and v, then we will have $\frac{\partial}{\partial \bar{z}} f(z) = 0$ for $f = u + iv$.

Note that the factor $\frac{1}{2}$ in the definition of $\frac{\partial}{\partial \bar{z}}$ plays no role in this equivalence. □

3. Determine whether the following functions $f(z) = f(x + iy)$ are analytic:

(a) $f(z) = x^2 + y^2$

(b) $f(z) = x^2 - y^2$

(c) $f(z) = x^2 - y^2 + 2ixy$

Solution: (a) No. We may write $f(z) = |z|^2 = z\bar{z}$, so $\frac{\partial}{\partial \bar{z}} f(z) = z \neq 0$.

(b) No. Write $u(x, y) = x^2 - y^2$ and $v(x, y) = 0$. Then the Cauchy-Riemann equations are not satisfied, since $\frac{\partial u}{\partial x} = 2x \neq 0 = \frac{\partial v}{\partial y}$.

(c) Yes, as we may write $f(z) = z^2$ which satisfies $\frac{\partial}{\partial \bar{z}} f(z) = 0$ for all z. Hence the Cauchy-Riemann equations are satisfied. But also, the real and imaginary parts of f are continuous and have continuous partial derivatives (as they are polynomials), so by exercise 2.3 on page 17 of the text, f is analytic. □

4. Find the domain of convergence of the following power series:

$$\text{(a)} \sum_{n=0}^{\infty} (z - 3i)^{2n} \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}$$

Solution: (a) By the ratio test, the radius of convergence is 1 about the point $3i$, so the domain of convergence is $\{z : |z - 3i| < 1\}$.

(b) By the ratio test, the radius of convergence is ∞, so the domain of convergence is the entire complex plane. □

5. Write a power series for the k^{th} derivative of

$$\sum_{n=0}^{\infty} (-1)^n z^n,$$

for all $k = 1, 2, \ldots$, and determine the domain of convergence. What functions do these power series represent?

Solution: The power series about $z = 0$ for the k^{th} derivative is

$$\sum_{n=k}^{\infty} (-1)^n (n) \cdots (n - k + 1) z^{n-k} = \sum_{n=k}^{\infty} (-1)^n \frac{n!}{(n-k)!} z^{n-k},$$

for all $k = 1, 2, \ldots$. The domain of convergence is $\{|z| < 1\}$ in all cases. These power series represent the functions

$$\frac{d^k}{dz^k} \left[\frac{1}{1+z} \right] = \frac{(-1)^k k!}{(1+z)^{k+1}}$$

on the domain of convergence. □

6. Determine, with proof, whether the following series converge uniformly on the domain $|z| < 1$:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{z^n}{n^2}; \qquad \text{(b)} \sum_{n=0}^{\infty} z^n.$$

Solution: (a) Yes. This series satisfies the Weierstrass M-test with constants $M_n = 1/n^2$, and everyone knows that $\sum 1/n^2 = \pi^2/6 < \infty$.

(b) No. The series diverges at $z = 1$, suggesting nonuniform convergence near there. For proof, for any real $0 < z < 1$ and any two integers $0 < P < Q$ we may compute

$$\left|\sum_{n=P}^{Q-1} z^n\right| = \sum_{n=P}^{Q-1} z^n = \sum_{n=0}^{Q-1} z^n - \sum_{n=0}^{P-1} z^n = \frac{z^P - z^Q}{1-z}.$$

To have uniform convergence, it is necessary that the rightmost expression can be made arbitrarily small for all $|z| < 1$ and any $Q > P$ simply by choosing large enough P. However, given any fixed P we observe that

$$\lim_{z\to 1-}\left(\lim_{Q\to\infty} \frac{z^P - z^Q}{1-z}\right) = \lim_{z\to 1-} \frac{z^P}{1-z} = +\infty,$$

so for every P there will always be some combination of z near 1 and big Q that yields big $\left|\sum_{n=P}^{Q-1} z^n\right|$. Hence the convergence of $\sum z^n$ cannot be uniform on $[0, 1)$. □