Ma 416: Complex Variables
Solutions to Homework Assignment 5

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Due Thursday, October 6th, 2005

Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 2, sections 8A–8H.

1. Classify the following singularities as removable, poles, or essential. If the singularity is a pole, state its order.

   (a) $1/(e^{z^2} - 1)$ at $z = 0$ (b) $e^{1/z}$ at $z = 0$ (c) $z/\sin z$ at $z = 0$

   Solution: (a) Pole of order 2, since $e^{z^2} - 1 = z^2 + O(z^4)$ as $z \to 0$.

   (b) Essential singularity, since $e^{1/z_n} = 1$ for $z_n = 1/2n\pi i \to 0$ but $e^{1/z_n} = -1$ for $z_n = 1/(2n+1)\pi i \to 0$ as $n \to \infty$. Hence the function can have no limit as $z \to 0$.

   (c) Removable singularity, since $\lim_{z\to 0}[z/\sin z] = 1/\lim_{z\to 0}[(\sin z)/z] = 1$. □

2. Find the residues of the following functions at the indicated points.

   (a) $1/(e^z - 1)$ at $z = 0$ (b) $z^4/(z - \frac{1}{6}z^3 - \sin z)$ at $z = 0$

   (c) $(z^2 + 1)/z^4 - 1)$ at $z = 1$ and $z = i$.

   Solution: (a) Since this is a simple pole, find the residue as follows:

$$\lim_{z\to 0} \frac{(z-0)}{e^z - 1} = \lim_{z\to 0} \frac{z}{z + O(z^2)} = \lim_{z\to 0} \frac{1}{1 + O(z)} = 1.$$

   (b) Note that $z - \frac{1}{6}z^3 - \sin z = -\frac{1}{120}z^5 + O(z^7)$ at $z = 0$. Hence $z^4/(z - \frac{1}{6}z^3 - \sin z)$ has a simple pole at $z = 0$. Find the residue as follows:

$$\lim_{z\to 0} \frac{(z-0)z^4}{z - \frac{1}{6}z^3 - \sin z} = \lim_{z\to 0} \frac{z^5}{-\frac{1}{120}z^5 + O(z^7)} = \lim_{z\to 0} \frac{-120}{1 + O(z^2)} = -120.$$

   (c) Factor the numerator and denominator polynomials into

$$\frac{(z-i)(z+i)}{(z-i)(z+i)(z-1)(z+1)} = \frac{1}{(z-1)(z+1)}.$$

   Hence $z = i$ (also $z = -i$) is a removable singularity, so the residue there is 0. However, $z = 1$ is a simple pole with residue $\lim_{z\to 1} \frac{(z-1)}{(z-1)(z+1)} = \frac{1}{2}$. □

3. Find the residue of the function $f(z) = 1/\sinh^2 z$ at $z = 0$.

Solution: Since $\sinh z = O(z)$ as $z \to 0$, we see that $z = 0$ is a pole of order 2 for $1/\sinh^2 z$. Find the residue as follows, setting $n = 2$ in the formula on page 73 of our text:

$$\begin{aligned}
\lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{(n-1)} [f(z)(z-z_0)^n] &= \lim_{z\to 0} \frac{d}{dz}\left[\frac{z^2}{\sinh^2 z}\right] \\
&= \lim_{z\to 0} \frac{2z\sinh z - 2z^2\cosh z}{\sinh^3 z} \\
&= 2\lim_{z\to 0}\left[\frac{z}{\sinh z}\right]\left[\frac{\sinh z - z\cosh z}{\sinh^2 z}\right]
\end{aligned}$$

L'Hôpital's rule allows us to evaluate the factor limits:

$$\lim_{z\to 0}\left[\frac{z}{\sinh z}\right] = \lim_{z\to 0}\left[\frac{1}{\cosh z}\right] = 1,$$

and

$$\lim_{z\to 0}\left[\frac{\sinh z - z\cosh z}{\sinh^2 z}\right] = \lim_{z\to 0}\left[\frac{z\sinh z}{2\sinh z\cosh z}\right] = 0.$$

Hence the residue is 0. □