# Ma 416: Complex Variables
# Solutions to Homework Assignment 6

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Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 2, sections 9A–11C.

1. Evaluate the definite integral $\int_0^{2\pi}(2+\sin\theta)^{-2}\,d\theta$.

   **Solution:** Substitute $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta \Rightarrow d\theta = (1/iz)\,dz$, and $\sin\theta = (z-1/z)/2i$ to get

$$\int_0^{2\pi}\frac{d\theta}{(2+\sin\theta)^2} = \oint_C \frac{dz}{iz\left(2+\frac{z-1/z}{2i}\right)^2} = \oint_C \frac{4iz\,dz}{(z^2+4iz-1)^2} = \oint_C \frac{4iz\,dz}{(z-z_+)^2(z-z_-)^2},$$

   where $C$ is the positively oriented unit circle $\{z = e^{i\theta} : 0 \le \theta \le 2\pi\}$, and $z_\pm = (-2\pm\sqrt{3})i$ are the roots of the denominator quadratic polynomial.

   Since $z_+ = (-2+\sqrt{3})i$ lies in the region $\{|z|<1\}$ enclosed by $C$, it is a pole of order 2 for the integrand $f(z) \stackrel{\text{def}}{=} 4iz(z-z_+)^{-2}(z-z_-)^{-2}$. The enclosed region contains no other poles of $f$, since $|z_-| = 2+\sqrt{3} > 1$. By the residue theorem, the integral equals $2\pi i\mathrm{Res}(z_+)$, where $\mathrm{Res}(z_+)$ is the residue of $f$ at $z_+$. Evaluate this residue with Cauchy's formula on page 73 of our text, with $n = 2$:

$$\begin{aligned}\mathrm{Res}(z_+) &= \lim_{z\to z_+}\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{(n-1)}[f(z)(z-z_+)^n] &= \lim_{z\to z_+}\frac{d}{dz}\left[\frac{4iz}{(z-z_-)^2}\right]\\ &= \lim_{z\to z_+}\frac{-4i(z+z_-)}{(z-z_-)^3} = \frac{-4i(z_++z_-)}{(z_+-z_-)^3} &= \frac{2}{3\sqrt{3}i}.\end{aligned}$$

   Hence $2\pi i\mathrm{Res}(z_+) = \dfrac{4\pi}{3\sqrt{3}}$ is the value of the original integral. □

2. Evaluate the improper integral $\int_0^\infty (x^4+1)^{-1}\,dx$.

   **Solution:** Write $z^4+1 = (z^2+i)(z^2-i) = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$, where $z_1 = (1+i)/\sqrt{2}$, $z_2 = (-1+i)/\sqrt{2}$, $z_3 = (-1-i)/\sqrt{2}$, and $z_4 = (1-i)/\sqrt{2}$ are the four $4^{\text{th}}$ roots of $-1$.

   Next, fix big $R > \sqrt{2}$, let $C = C_h \cup C_a \cup C_v$ be the simple closed piecewise smooth quarter-circle contour in the first quadrant consisting of the horizontal line segment $C_h = \{z = x : x \in [0,R]\}$, the quarter-arc $C_a = \{Re^{i\theta} : 0 \le \theta \le \pi/2\}$, and the vertical line segment $C_v = \{z = (R-y)i : y \in [0,R]\}$. As parametrized, $C$ has positive orientation.

The region enclosed by $C$ will contain the simple pole $z_1$ of $f(z) \stackrel{\text{def}}{=} (z^4+1)^{-1}$ and no other singular points of $f$. Hence by the residue theorem, $\oint_C f(z)\,dz = 2\pi i\,\text{Res}(z_1)$, where $\text{Res}(z_1)$ is the residue of $f$ at $z_1$. But since $z_1$ is a simple pole,

$$\text{Res}(z_1) = \lim_{z\to z_1}(z-z_1)f(z) = [(z_1-z_2)(z_1-z_3)(z_1-z_4)]^{-1} = \frac{1}{2i(1+i)\sqrt{2}}.$$

Thus $\oint_C f(z)\,dz = \dfrac{\pi}{(1+i)\sqrt{2}}$ for any $R > \sqrt{2}$.

Also, $\oint_C f(z)\,dz = \int_{C_h} f(z)\,dz + \int_{C_a} f(z)\,dz + \oint_{C_v} f(z)\,dz$, where

$$\begin{aligned}
\oint_{C_h} f(z)\,dz &= \int_0^R \frac{dx}{x^4+1}; \\
\oint_{C_v} f(z)\,dz &= \int_0^R \frac{di(R-y)}{(R-y)^4+1} \;=\; -i\int_0^R \frac{dt}{t^4+1}, \qquad \text{after } y \leftarrow (R-t); \\
\oint_{C_a} f(z)\,dz &= \int_0^{\pi/2} \frac{iRe^{i\theta}\,d\theta}{R^4e^{4i\theta}+1} \\
\Rightarrow \left|\oint_{C_a} f(z)\,dz\right| &\le \frac{\pi}{2}\,\frac{R}{R^4-1} \to 0 \quad \text{as } R\to\infty.
\end{aligned}$$

The integral over the arc $C_a$ must therefore vanish as $R\to\infty$, leaving

$$\lim_{R\to\infty}\oint_C f(z)\,dz = (1-i)\int_0^\infty \frac{dx}{x^4+1}.$$

The limit $\pi/[(1+i)\sqrt{2}]$ on the left-hand side is attained as soon as $R > \sqrt{2}$. Hence

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{(1-i)(1+i)\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

$\square$

3. Evaluate the principal value integral $\text{PV}\int_{-\infty}^{\infty}(x^3-1)^{-1}\,dx$.

**Solution:** Note that $f(z) \stackrel{\text{def}}{=} (z^3-1)^{-1}$ has simple poles at the three cube roots of 1, namely $z_1 = e^{2\pi i/3} = (-1+i\sqrt{3})/2$, $z_2 = e^{4\pi i/3} = (-1-i\sqrt{3})/2$, and $z_3 = e^{6\pi i/3} = 1$.

Fix $R > 2$ and $0 < \epsilon < 1$ and consider the piecewise differentiable curve $C = C_1 \cup C_\epsilon \cup C_2 \cup C_R$ defined by

$$\begin{aligned}
C_1 &= \{z = x : -R \le x \le 1-\epsilon\}; \\
C_\epsilon &= \{z = 1-\epsilon e^{-i\theta} : 0 \le \theta \le \pi\}; \\
C_2 &= \{z = x : 1+\epsilon \le x \le R\}; \\
C_R &= \{z = Re^{i\theta} : 0 \le \theta \le \pi\}.
\end{aligned}$$

Then $C$ is a simple closed curve that encloses one pole of $f$ (namely $z_1$), skirts within $\epsilon$ of another ($z_3 = 1$), and completely excludes the third ($z_2$). Note that $C$ is positively oriented. By the residue theorem,

$$\oint_C f(z) = 2\pi i\text{Res}(z_1) = 2\pi i\lim_{z\to z_1}(z-z_1)f(z) = \frac{2\pi i}{(z_1-z_2)(z_1-z_3)} = \frac{4\pi}{3(i-\sqrt{3})}.$$

The four path integrals comprising $\oint_C f(z)\,dz$ may be written as follows:

$$\begin{aligned}
\int_{C_1\cup C_2} f(z)\,dz &= \int_{-R}^{1-\epsilon} \frac{dx}{x^3-1} + \int_{1+\epsilon}^{R} \frac{dx}{x^3-1};\\
\int_{C_\epsilon} f(z)\,dz &= \int_0^\pi \frac{i\epsilon e^{-i\theta}\,d\theta}{[1-\epsilon e^{-i\theta}]^3-1} = \int_0^\pi \frac{i\,d\theta}{-3+3\epsilon e^{-i\theta}-\epsilon^2 e^{-2i\theta}};\\
\int_{C_R} f(z)\,dz &= \int_0^\pi \frac{iRe^{i\theta}\,d\theta}{R^3e^{3i\theta}-1}.
\end{aligned}$$

Evidently $\int_{C_\epsilon} f(z)\,dz \to -i\pi/3$ as $\epsilon \to 0$, and $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

As $\epsilon \to 0$ and $R \to \infty$, the path integral $\int_{C_1\cup C_2} f(z)\,dz$ converges to the principal value we are seeking. Combining these calculations gives

$$\mathrm{PV}\int_{-\infty}^{\infty} (x^3-1)^{-1}\,dx = \frac{4\pi}{3(i-\sqrt{3})} - \frac{-i\pi}{3} - 0 = \frac{-\pi}{\sqrt{3}}$$

Note that the negative values of the integrand in $-\infty < x < 1$ outweigh the positive values in $1 < x < \infty$. □

4. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{x/n}}{1-e^x}\,dx,$$

for $n = 1, 2, 3, \ldots$. (Hint: use rectangular contours of fixed height $2\pi$ with one side lying on the $x$-axis.)

**Solution:** Following the hint, fix $R > 2$ and $0 < \epsilon < 1$ and let $C = C_B \cup C_{\epsilon B} \cup C_U \cup C_T \cup C_{\epsilon T} \cup C_D$ be the piecewise differentiable rectangular contour with two nibbles near $z = 0$ and $z = 2\pi i$ defined as follows:

$$\begin{aligned}
C_B &= \{z = x : -R \le x \le -\epsilon \text{ and } \epsilon \le x \le R\};\\
C_T &= \{z = 2\pi i - x : -R \le x \le -\epsilon \text{ and } \epsilon \le x \le R\};\\
C_U &= \{z = R + iy : 0 \le y \le 2\pi\};\\
C_D &= \{z = -R + i(2\pi - y) : 0 \le y \le 2\pi\};\\
C_{\epsilon B} &= \{z = -\epsilon e^{-i\theta} : 0 \le \theta \le \pi\};\\
C_{\epsilon T} &= \{z = 2\pi i + \epsilon e^{-i\theta} : 0 \le \theta \le \pi\}.
\end{aligned}$$

It may be seen that $C$ is a simple closed curve.

For each integer $n > 1$, the function $f(z) = e^{z/n}/(1-e^z)$ only has simple poles at $z_k = 2k\pi i$ for any integer $k$, so it has no singular points in the region enclosed by $C$. The nibbles $C_{\epsilon B}$ and $C_{\epsilon T}$ avoid the poles $z_0 = 0$ and $z_1 = 2\pi i$, respectively. By the residue theorem, $\oint_C f(z)\,dz = 0$.

Now

$$\int_{C_B} f(z)\,dz = \int_{-R}^{-\epsilon} \frac{e^{x/n}\,dx}{1-e^x} + \int_{\epsilon}^{R} \frac{e^{x/n}\,dx}{1-e^x} \stackrel{\text{def}}{=} \int_I \frac{e^{x/n}\,dx}{1-e^x},$$

where $I = [-R,-\epsilon] \cup [\epsilon, R]$. This tends to our desired integral as $R \to \infty$ and $\epsilon \to 0$. But also,

$$\int_{C_T} f(z)\,dz = \int_I \frac{e^{[2\pi i - x]/n}\,d[2\pi i - x]}{1-e^{[2\pi i - x]}} = e^{2\pi i/n}\int_I \frac{e^{-x/n}\,d(-x)}{1-e^{-x}} = -e^{2\pi i/n}\int_I \frac{e^{x/n}\,dx}{1-e^x},$$

after the substitution $x \leftarrow -x$. Thus

$$\int_{C_B \cup C_T} f(z)\,dz = (1 - e^{2\pi i/n}) \int_I \frac{e^{x/n}\,dx}{1-e^x}.$$

The values for $\int_{C_{\epsilon B}} f(z)\,dz$ and $\int_{C_{\epsilon T}} f(z)\,dz$ are computed as in Exercise 9.5 on pages 84–85 of our text. They are "partial residues," exactly half the residues at $z_0$ and $z_1$, respectively, and negative because the poles are outside the contour $C$:

$$\begin{aligned}
\int_{C_{\epsilon B}} f(z)\,dz &= -\frac{1}{2}[2\pi i]\mathrm{Res}(z_0) &= \pi i; \\
\int_{C_{\epsilon T}} f(z)\,dz &= -\frac{1}{2}[2\pi i]\mathrm{Res}(z_1) &= e^{2\pi/n}\pi i;
\end{aligned}$$

Here we have computed the residues in the usual way:

$$\begin{aligned}
\mathrm{Res}(z_0) &= \mathrm{Res}(0) = \lim_{z\to 0} \frac{ze^{z/n}}{1-e^z} &= -1; \\
\mathrm{Res}(z_1) &= \mathrm{Res}(2\pi i) = \lim_{z\to 2\pi i} \frac{(z-2\pi i)e^{z/n}}{1-e^z} &= -e^{2\pi i/n}
\end{aligned}$$

The remaining contour integrals may be estimated:

$$\begin{aligned}
\left|\int_{C_U} f(z)\,dz\right| &\le \int_0^{2\pi} \frac{e^{R/n}|e^{iy/n}|\,dy}{|1-e^Re^{iy}|} \le \frac{2\pi e^{R/n}}{e^R - 1} \to 0, \quad \text{as } R\to\infty; \\
\left|\int_{C_D} f(z)\,dz\right| &\le \int_{2\pi}^0 \frac{e^{-R/n}|e^{iy/n}|\,dy}{|1-e^{-R}e^{iy}|} \le \frac{2\pi e^{-R/n}}{1-e^{-2}} \to 0, \quad \text{as } R\to\infty.
\end{aligned}$$

Combining these computations gives the value of the integral:

$$\int_{-\infty}^{\infty} \frac{e^{x/n}\,dx}{1-e^x} = \frac{-(1+e^{2\pi i/n})\pi i}{1-e^{2\pi i/n}} = \frac{(e^{-\pi i/n}+e^{\pi i/n})\pi i}{-(e^{-\pi i/n}-e^{\pi i/n})} = \pi\cot(\pi/n).$$

Note that this integral must be interpreted in the principal value sense. □

5. For each relation below, find all the complex numbers $z$ satisfying it:

   (a) $z^n = 1$ for a fixed $n \in \{2,3,4,\ldots\}$

   (b) $e^z = -e$

   (c) $e^{\sqrt{z}} = i$

   (d) $\tan z = 1 + i$.

   **Solution:** (a) $z \in \{e^{2k\pi i/n} : k = 1,2,\ldots,n\}$. There are $n$ distinct values in this set, the $n$ $n^{\text{th}}$ roots of unity, equidistributed around the unit circle.

   (b) Since $-e = e^{i\pi}e^1 = e^{1+i\pi}$ and the exponential function is $2\pi i$-periodic, the solution set is $\{1+\pi i + 2k\pi i = 1 + (2k+1)\pi i : k \in \mathbf{Z}\}$.

   (c) Since $e^{i\pi/2} = i$ and the exponential function is $2\pi i$-periodic, the solution set is $\{z^2 : z = i\pi/2 + 2k\pi i = (2k+\frac{1}{2})\pi i : k \in \mathbf{Z}\} = \{-(2k+\frac{1}{2})^2\pi^2 : k \in \mathbf{Z}\}$.

(d) Use the idea in the solution to Exercise 10.11 on page 95 of our text:

$$
\begin{aligned}
1+i \;=\; \tan z \quad &= \quad \frac{\sin z}{\cos z} \;=\; \frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} \\
&\Rightarrow \quad (2-i)e^{iz} \;=\; ie^{-iz} \\
&\Rightarrow \quad e^{2iz} \;=\; \frac{i}{2-i} \;=\; -\frac{1}{5}+\frac{2}{5}i \;=\; \frac{1}{\sqrt{5}}\left(-\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i\right).
\end{aligned}
$$

The final expression on the right-hand side is in the form $re^{i\theta}$ with $r = 1/\sqrt{5}$ and $\theta \in \mathbf{R}$ satisfying $\cos\theta = -1/\sqrt{5}$, $\sin\theta = 2/\sqrt{5}$. From this we conclude that

$$
\begin{aligned}
\Re(2iz) \quad &= \quad -2\Im(z) \;=\; \ln(r) \;\approx\; -0.8047 \\
&\Rightarrow \quad \Im(z) \;\approx\; 1.6094; \\
\Im(2iz) \quad &= \quad 2\Re(z) \;=\; \theta \;\in\; \{\arctan(-2)+\pi+2k\pi : k \in \mathbf{Z}\} \\
&\Rightarrow \quad \Re(z) \;\in\; \{1.0172 + k\pi : k \in \mathbf{Z}\},
\end{aligned}
$$

using the approximation $\arctan(-2) \approx -1.1071$, the principal value of arctangent which lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We must add $\pi$ to this principal value to get an angle $\theta \approx 2.0344$ in the second quadrant satisfying $\sin\theta > 0$, $\cos\theta < 0$. Hence the solution set may be written approximately as $\{1.0172 + 1.6094i + k\pi : k \in \mathbf{Z}\}$. □

6. Evaluate the integral

$$\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}}.$$

**Solution:** This is a special case of Exercise 11.1(a) on page 97 of the textbook:

$$\int_0^\infty \frac{x^\lambda \, dx}{a^2+x^2}, \qquad a = 1, \lambda = -\frac{1}{2}.$$

We follow the model solution on page 294 of the textbook, substituting $x^2 \leftarrow a^2 t = t$ to obtain the equivalent integral

$$\frac{1}{2}a^{\lambda-1}\int_0^\infty \frac{t^{-1+(1+\lambda)/2}\, dt}{1+t} = \frac{1}{2}\int_0^\infty \frac{t^{-3/4}\, dt}{1+t}, = \frac{1}{2}\int_0^\infty \frac{t^{\frac{1}{4}-1}\, dt}{1+t},$$

which evaluates to

$$\frac{1}{2}a^{\lambda-1}I(\frac{1}{4}) = \frac{1}{2}\frac{\pi}{\sin\frac{1}{4}\pi} = \frac{\pi}{\sqrt{2}},$$

using the notation and technique in Section 11A, pages 95–97 of the textbook. □