Ma 416: Complex Variables
Solutions to Homework Assignment 9

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Due Thursday, November 10th, 2005

Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 2, sections 16A–16C.

1. Suppose f is analytic on the closed unit disk, $f(0) = 0$, and $|f(z)| \leq |e^z|$ whenever $|z| = 1$. How big can $f((1+i)/2)$ be?

   Solution: Note that $|f(z)| \leq \max\{|e^z| : |z| = 1\} \leq \max\{|e^{\cos\theta}| : \theta \in [0, 2\pi]\} = e$. By the Schwartz lemma, $|f(z)| \leq e|z|$ for any z in the closed unit disk. Therefore, $|f((1+i)/2)| \leq e|(1+i)/2| = e/\sqrt{2}$. □

2. Prove Schwarz's lemma for a disk of radius R: If f is analytic on a closed disk D of radius R centered at z_0, $f(z_0) = 0$, and $|f(z)| \leq M$ on the boundary circle of D, then $|f(z)| \leq |z - z_0|M/R$ for each z inside D, with equality holding at some interior point z if and only if $f(z) = e^{ic}(z - z_0)$ for some constant $c \in \mathbf{R}$.

   Solution: This follows from Schwarz's lemma for a disk of radius 1, as stated on page 133 of our text, applied to the analytic function $g(z) \stackrel{\text{def}}{=} f(Rz + z_0)$. Evidently, g is analytic in a closed disk Δ of radius 1 centered at 0 and satisfies $g(0) = f(z_0) = 0$. If $|f(z)| \leq M$ on the boundary circle of D, then $|g(z)| \leq M$ on the boundary circle of Δ. Apply the radius-1 Schwarz lemma to conclude that

   $$|f(Rz + z_0)| = |g(z)| \leq M|z|, \qquad \Rightarrow |f(w)| \leq \frac{M}{R}|w - z_0|,$$

   after the substitution $z \leftarrow (w - z_0)/R$. Equality holds if and only if $g(z) = Me^{ic}z$ if and only if $f(w) = Me^{ic}(w - z_0)/R$ for some $c \in \mathbf{R}$.

   It remains to prove the radius-1, center-0 Schwarz lemma for g. So let $h(z) = g(z)/z$; then h is analytic in the closed unit disk Δ since it has a removable singularity at $z = 0$. By the Maximum Modulus Theorem, $|h(z)|$ attains its maximum somewhere on the boundary unit circle of Δ. Call this maximum absolute value M; then

   $$|h(z)| \leq M \Rightarrow |g(z)| \leq M|z|,$$

   for all $z \in D$. □

3. Use the radius-R Schwarz lemma of Problem 2 to prove Liouville's theorem. (Hint: apply the lemma to $f(z) - f(0)$.)

Solution: If f is entire analytic and bounded, then so is $g(z) \stackrel{\text{def}}{=} f(z) - f(0)$: if M is an upper bound for f, so $|f(z)| \leq M$ for all $z \in \mathbf{C}$, then

$$|g(z)| \leq |f(z)| + |f(0)| \leq M + |f(0)| \stackrel{\text{def}}{=} M' < \infty,$$

and of course $g'(z) = f'(z)$ for any $z \in \mathbf{C}$. Note that $g(0) = 0$.

For any $R > 0$, let D_R be the disk of radius R centered at 0. By the radius-R Schwarz lemma, for any $z \in D_R$,

$$|g(z) - g(0)| = |g(z)| \leq |z|\frac{M'}{R} \quad \Rightarrow 0 \leq \left|\frac{g(z) - g(0)}{z}\right| \leq \frac{M'}{R}.$$

But as $R \to \infty$, the right-hand side tends to 0. We conclude that $g(z) = g(0) = 0$ for every $z \neq 0$. But then $f(z) = f(0)$ for every $z \neq 0$, so f must be constant. □

4. Prove that an entire function whose real part is bounded must be constant. (Hint: apply Liouville's theorem to the function e^f.)

Solution: Suppose f is entire analytic and $|\Re f(z)| \leq M$ for all $z \in \mathbf{C}$. Then $g(z) \stackrel{\text{def}}{=} e^{f(z)}$ is entire analytic as well, since $g'(z) = e^{f(z)} f'(z)$ by the chain rule. But

$$|g(z)| = |e^{\Re f(z)}| \times |e^{i\Im f(z)}| = |e^{\Re f(z)}| \leq e^M,$$

so g is bounded. By Liouville's theorem, g must be constant. Since f is continuous and $e^f = g$, we conclude that f must be constant. □

5. Suppose that f is analytic on the closed unit disk, $f(0) = 0$, and $|\Re f(z)| \leq |e^z|$ for $|z| < 1$. Can $f((1+i)/2)$ be 18?

Solution: Note that $A(r) \stackrel{\text{def}}{=} \max\{|\Re f(z)| : |z| = r\} = e^r$ for each $0 \leq r < 1$. Using Carathéodory's inequality for f gives:

$$|f(re^{i\theta})| \leq |f(0)| + \frac{2r}{1-r}[A(1) - \Re f(0)] \leq \frac{2r}{1-r}[e+1],$$

since $f(0) = 0$, $A(1) = e$, and $\Re f(0) \geq -A(0) = -1$. Writing $(1+i)/2 = \frac{1}{\sqrt{2}}e^{i\pi/4}$ identifies $r = 1/\sqrt{2}$, giving the estimate

$$|f((1+i)/2)| \leq \frac{2/\sqrt{2}}{1 - 1/\sqrt{2}}[e+1] = \frac{2}{\sqrt{2}-1}[e+1] \approx 17.95,$$

so $f((1+i)/2)$ cannot be 18. □