# Ma 416: Complex Variables
# Solutions to Homework Assignment 11

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Due Thursday, December 1st, 2005

Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 4, sections 19A–20F.

1. Find an analytic function $f$ whose real part is $\Re f(x+iy) = x^3y - xy^3$.

   **Solution:** Use $f(z) = z^4/(4i)$, so

$$\begin{aligned} f(x+iy) &= \frac{(x+iy)^4}{4i} = \frac{x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4}{4i} \\ &= x^3y - xy^3 - i\left(\frac{x^4 - 6x^2y^2 + y^4}{4}\right). \end{aligned}$$

   This $f$ is a polynomial, hence it is an an entire analytic function. □

2. Find the conjugate harmonic function of $g(x,y) = e^{-y}\cos x$.

   **Solution:** Observe that $g$ is the real part of the entire analytic function $f(z) = e^{iz}$:

$$f(x+iy) = e^{i(x+iy)} = e^{ix-y} = e^{-y}\cos x + ie^{-y}\sin x.$$

   Hence its harmonic conjugate is the imaginary part: $\tilde{g}(x,y) = e^{-y}\sin x$. □

3. Is the function $h(x,y) = x^2 + y^2$ the imaginary part of some function $f(x+iy)$ analytic in the unit disk in $\mathbf{C}$?

   **Solution:** No; $h$ is not a harmonic function, since $\Delta h(x,y) = 2+2 = 4 \neq 0$ for any $(x,y)$. Hence it cannot be the real or imaginary part of an analytic function. □

4. For $0 \le r < 1$ and $0 \le \phi \le 2\pi$, define the function

$$I(r,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,d\theta}{1 + r^2 - 2r\cos(\theta - \phi)}.$$

   (This is the integral of the Poisson kernel $P(r, \theta - \phi)$.)

   (a) Show that $I(r,\phi)$ does not depend on $\phi$. (Hint: substitute $\theta \leftarrow \theta' + \phi$.)

(b) Show that $I(r,\phi)$ does not depend on $r$. (Hint: put $a = 2r/(1+r^2)$, observe that $(1-r^2)/(1+r^2) = \sqrt{1-a^2}$, and look up $\int_{-\pi}^{\pi} d\theta/(1-a\cos\theta) = 2\pi/\sqrt{1-a^2}$ in a table of integrals.)

(c) Show that $I(r,\phi) = 1$ for all $0 \le r < 1$ and $0 \le \phi \le 2\pi$ by evaluating $I(0,0) = 1$ and using parts (a) and (b).

(d) Conclude that if $u = u(x,y)$ is a harmonic function on the unit disk $D = \{x^2+y^2 \le 1\}$, and $u(x,y) = K$ for all $x^2+y^2 = 1$, then $u(x,y) = K$ for all $(x,y) \in D$.

**Solution:** (a) Since $[-\pi,\pi]$ and $[-\pi+\phi,\pi+\phi]$ are both period intervals for $\cos(\theta-\phi)$ for any $\phi$, we may shift and use the substitution $\theta \leftarrow \theta' + \phi$ with $d\theta = d\theta'$ to get

$$\begin{aligned} I(r,\phi) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,d\theta}{1+r^2-2r\cos(\theta-\phi)} \\ &= \frac{1}{2\pi}\int_{-\pi+\phi}^{\pi+\phi} \frac{(1-r^2)\,d\theta}{1+r^2-2r\cos(\theta-\phi)} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)\,d\theta'}{1+r^2-2r\cos(\theta'-0)} \quad = \quad I(r,0). \end{aligned}$$

But this is true for all $\phi$, so $I(r,\phi)$ must be independent of $\phi$.

(b) First use part (a) to conclude that it suffices to show that $I(r,\phi) = I(r,0)$ is independent of $r$. Following the hint, observe that if $0 \le r < 1$, then $a \stackrel{\text{def}}{=} 2r/(1+r^2)$ also satisfies $0 \le a < 1$, and we have

$$\sqrt{1-a^2} = \sqrt{\frac{[1+r^2]^2-[2r]^2}{[1+r^2]^2}} = \sqrt{\frac{[1-r^2]^2}{[1+r^2]^2}} = \frac{1-r^2}{1+r^2}.$$

Thus, dividing numerator and denominator of the integrand of $I(r,0)$ by $1+r^2 > 0$ gives

$$I(r,\phi) = I(r,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sqrt{1-a^2}\,d\theta}{1-a\cos(\theta)}.$$

Using a table of integrals, we evaluate

$$\int_{-\pi}^{\pi} \frac{d\theta}{1-a\cos(\theta)} = \frac{2\pi}{\sqrt{1-a^2}},$$

for any $a$ with $a^2 < 1$, using in this case formula 3.613 on page 409 of Gradshteyn and Ryzhik (fifth edition, 1994, ISBN 0-12-294755-X). But then

$$I(r,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sqrt{1-a^2}\,d\theta}{1-a\cos(\theta)} = 1,$$

independently of $a$ and thus of $r$.

(c) In addition to the direct evaluation in part (b), we may use the constancy of $I(r,\phi)$ to evaluate it at an easy point:

$$I(r,\phi) = I(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1\,d\theta}{1-0} = 1.$$

(d) If the real part of an analytic function $f(z)$ on $D$ takes the constant value $\Re f(e^{i\theta}) = K$ on the boundary of $D$, then it must have constant real part $K$ inside $D$, since

$$\Re f(re^{i\phi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re f(e^{i\theta}) \frac{(1-r^2)\, d\theta}{1+r^2-2r\cos(\theta-\phi)} = K\, I(r,\phi) = K.$$

But if $u$ is any harmonic function on $D$, then $u = \Re f$ for some analytic $f$, so we conclude that $u(x,y) = K$ for all $(x,y) \in D$. $\square$

5. Show directly that $u(x,y) = x^2 - y^2$ satisfies the averaging property: if $R > 0$, $C_R = \{r(\theta) = (x_0 + R\cos\theta, y_0 + R\sin\theta) : 0 \le \theta \le 2\pi\}$, and $ds = \|r'(\theta)\|\, d\theta$ is the arc length differential on $C_R$, then

$$\oint_{C_R} u(x,y)\, ds = 2\pi R\, u(x_0, y_0).$$

How can Cauchy's integral formula be used to derive the same results?

**Solution:** Let $x(\theta) = x_0 + R\cos\theta$ and $y(\theta) = y_0 + R\sin\theta$ be the coordinate functions for the given parameterization of $C_R$. Then $r(\theta) = (x(\theta), y(\theta))$, $r'(\theta) = R(-\sin\theta, \cos\theta)$, and $\|r'(\theta)\| = R$ for all $\theta \in [0, 2\pi]$, so $ds = R\, d\theta$.

Now evaluate

$$\begin{aligned} u(x(\theta), y(\theta)) - u(x_0, y_0) &= 2R(x_0\cos\theta - y_0\sin\theta) + R^2(\cos^2\theta - \sin^2\theta) \\ &= 2R(x_0\cos\theta - y_0\sin\theta) + R^2\cos(2\theta), \end{aligned}$$

so

$$\oint_{C_r} [u(x,y) - u(x_0,y_0)]\, ds = \int_0^{2\pi} [2R(x_0\cos\theta - y_0\sin\theta) + R^2\cos(2\theta)] R\, d\theta = 0,$$

as $\cos\theta$, $\sin\theta$, and $\cos(2\theta)$ all have integral zero over the period interval $[0, 2\pi]$. Thus

$$\oint_{C_r} u(x,y)\, ds = \oint_{C_r} u(x_0,y_0)\, ds.$$

Finally, compute

$$\oint_{C_r} u(x_0,y_0)\, ds = Ru(x_0,y_0)\int_0^{2\pi} 1\, d\theta = 2\pi R\, u(x_0,y_0).$$

Alternatively, use the fact that a harmonic function is the real part of an analytic function, after checking that $u(x,y)$ is harmonic by testing the Laplacian:

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

Since $\Delta u(x,y) = 0$ everywhere, and the polynomial $u$ is evidently differentiable everywhere, we conclude that $u$ is harmonic. By Cauchy's integral theorem, $u$ satisfies the mean value property. $\square$