Ma 416: Complex Variables
Solutions to Homework Assignment 12

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Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 4, sections 21A–23B and 25A-25E.

1. Suppose that $u = u(x, y)$ is continuous on the closed unit disk $D = \{(x, y) : x^2+y^2 \le 1\}$ and u is twice continuously differentiable with $\Delta u(x, y) = 0$ inside D. Find a series representing u for each of the following boundary conditions $u(\cos\theta, \sin\theta) = \psi(\theta)$, $-\pi \le \theta \le \pi$:

$$(a) \quad \psi(\theta) = \begin{cases} 1, & \text{if } \theta \in [-\pi/2, \pi/2]; \\ 0, & \text{otherwise,} \end{cases} \qquad (b) \quad \psi(\theta) = \sin\theta.$$

Solution: Let $\psi(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$ denote the Fourier series of the boundary function.

(a) First find the Fourier coefficients:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\phi(\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ik\theta}\,d\theta = \frac{1}{\pi k}\sin\frac{k\pi}{2} = \begin{cases} [\pi k]^{-1}, & k = 4j+1; \\ -[\pi k]^{-1}, & k = 4j-1; \\ 0, & \text{otherwise.} \end{cases}$$

Then the solution is

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta} = \sum_{j=-\infty}^{\infty} \left[\frac{r^{|4j+1|}}{\pi(4j+1)} e^{i(4j+1)\theta} - \frac{r^{|4j-1|}}{\pi(4j-1)} e^{i(4j-1)\theta} \right]$$

(b) Find the Fourier coefficients:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\phi(\theta)\,d\theta = \begin{cases} -i/2, & \text{if } k = 1; \\ i/2, & \text{if } k = -1; \\ 0, & \text{otherwise.} \end{cases}$$

That gives $\psi(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} = -\frac{i}{2}e^{i\theta} + \frac{i}{2}e^{-i\theta} = (e^{i\theta} - e^{-i\theta})/2i = \sin\theta$ as required. Then the solution inside the disk is

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta} = r\sin\theta.$$

This is the imaginary part of the entire analytic function $f(z) = z$. □

2. Suppose that $u(x, y)$ is continuous on the closed annulus $A = \{(x, y) : 1 \le x^2 + y^2 \le 4\}$ and u is twice continuously differentiable with $\Delta u(x, y) = 0$ inside A. Find a series representing u for each of the following boundary conditions $u(\cos\theta, \sin\theta) = \psi_1(\theta)$ and $u(2\cos\theta, 2\sin\theta) = \psi_2(\theta)$, $-\pi \le \theta \le \pi$:

$(a)\quad \psi_1(\theta) = 0;\ \psi_2(\theta) = |\theta|;\qquad\qquad (b)\quad \psi_1(\theta) = \cos\theta;\ \psi_2(\theta) = \sin\theta$

Solution: Let $\psi_1(\theta) = \sum_{k=-\infty}^{\infty} c_k^1 e^{ik\theta}$ and $\psi_2(\theta) = \sum_{k=-\infty}^{\infty} c_k^2 e^{ik\theta}$ denote the Fourier series of the boundary functions. Observe that the inner and outer radii of the annulus are $r_1 = 1$ and $r_2 = 2$, respectively.

Following Exercise 21.4, as solved on pp.316–317 of our text, the solution may be written as a linear combination of the separated components solving the r and θ parts of:

$$u(re^{i\theta}) = c\ln r + \sum_{k=-\infty}^{\infty} (a_k r^k + b_k r^{-k}) e^{ik\theta}.$$

Identifying the coefficients of $e^{ik\theta}$ with the Fourier coefficients of the boundary functions at $r = 1$ and $r = 2$ gives

$$\begin{aligned}
c_0^1 &= c\ln 1 + a_0 + b_0 = a_0 + b_0 \\
c_0^2 &= c\ln 2 + a_0 + b_0 \\
&\Rightarrow c = (c_0^2 - c_0^1)/\ln 2; \\
&\Rightarrow a_0 + b_0 = c_0^1; \\
c_k^1 &= a_k 1^k + b_k 1^{-k} = a_k + b_k \\
c_k^2 &= a_k 2^k + b_k 2^{-k} \\
&\Rightarrow a_k = (c_k^2 - 2^{-k} c_k^1)/(2^k - 2^{-k}) \\
&\Rightarrow b_k = (c_k^2 - 2^k c_k^1)/(2^{-k} - 2^k).
\end{aligned}$$

(a) First find the Fourier coefficients:

$$\begin{aligned}
c_k^1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\phi_1(\theta)\, d\theta = 0, \qquad \text{all } k \in \mathbf{Z}; \\
c_k^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\phi_2(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta| e^{-ik\theta}\, d\theta = \begin{cases} \pi/2, & k = 0; \\ -2/[\pi k^2], & k \text{ odd}; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

For c_k^2 we use the identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta| e^{-ik\theta}\, d\theta = \frac{1}{\pi}\int_0^{\pi} \theta\cos k\theta\, d\theta$$

and then integrate by parts.

Using the formulas for the separated expansion of u developed in the initial discussion gives

$$
\begin{aligned}
c &= \pi/(2\ln 2); \\
a_0 + b_0 &= 0; \\
a_k &= \begin{cases} 0, & \text{if } k \neq 0 \text{ is even;} \\ -2/[\pi k^2(2^k - 2^{-k})], & \text{if } k \text{ is odd;} \end{cases} \\
b_k &= \begin{cases} 0, & \text{if } k \neq 0 \text{ is even;} \\ 2/[\pi k^2(2^k - 2^{-k})], & \text{if } k \text{ is odd.} \end{cases}
\end{aligned}
$$

Hence the solution is

$$
u(re^{i\theta}) = \frac{\pi}{2\ln 2}\ln r - \frac{2}{\pi}\sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{r^k - r^{-k}}{\pi k^2(2^k - 2^{-k})} e^{ik\theta}.
$$

(b) Find the Fourier coefficients:

$$
\begin{aligned}
c_k^1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\phi_1(\theta)\,d\theta = \begin{cases} 1/2, & \text{if } k = 1 \text{ or } k = -1; \\ 0, & \text{otherwise, as in Problem 1(b).} \end{cases} \\
c_k^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\phi_2(\theta)\,d\theta = \begin{cases} -i/2, & \text{if } k = 1; \\ i/2, & \text{if } k = -1; \\ 0, & \text{otherwise, as in Problem 1(b);} \end{cases}
\end{aligned}
$$

Using the formulas for the separated expansion of u developed in the initial discussion gives

$$
\begin{aligned}
c &= 0; \\
a_0 + b_0 &= 0; \\
a_k &= \begin{cases} (2-i)/3, & \text{if } k = -1; \\ -(2i+1)/6, & \text{if } k = 1; \\ 0, & \text{otherwise;} \end{cases} \\
b_k &= \begin{cases} (2i-1)/6, & \text{if } k = -1; \\ (2+i)/3, & \text{if } k = 1; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Hence the solution is

$$
u(re^{i\theta}) = \left[\frac{2-i}{3r} + \frac{2i-1}{6}r\right]e^{-i\theta} + \left[\frac{2+i}{3r} - \frac{2i+1}{6}r\right]e^{i\theta}.
$$

□

3. Suppose that $f = f(z)$ is analytic and univalent in a region $D \subset \mathbf{C}$ and let $E = f(D) = \{f(z) : z \in D\}$ be its range. Write $u + iv = f(x+iy)$ and identify (u,v) with $u+iv$. Prove that if $\phi = \phi(u,v)$ is a harmonic function in E, then $\psi(x,y) \stackrel{\text{def}}{=} \phi(f(x+iy))$ is a harmonic function in D.

Solution: Write $u = u(x,y) = \Re f(x+iy)$ and $v = v(x,y) = \Im f(x+iy)$, so we have $\psi(x,y) = \phi(u(x,y), v(x,y))$. By the chain rule for functions of two real variables,

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x};\\
\frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial^2 \phi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 \phi}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 \phi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial \phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial \phi}{\partial v}\frac{\partial^2 v}{\partial x^2};\\
\frac{\partial \psi}{\partial y} &= \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y};\\
\frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial^2 \phi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 \phi}{\partial u \partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 \phi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial \phi}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial \phi}{\partial v}\frac{\partial^2 v}{\partial y^2};
\end{aligned}$$

Here we have used the equality of mixed partial derivatives for the harmonic function ϕ, namely $\frac{\partial^2 \phi}{\partial u \partial v} = \frac{\partial^2 \phi}{\partial v \partial u}$. We now add the second partial derivatives and rearrange the terms to get:

$$\begin{aligned}
\Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial^2 \phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial^2 \phi}{\partial v^2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] +\\
&+ 2\frac{\partial^2 \phi}{\partial u \partial v}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right] +\\
&+ \frac{\partial \phi}{\partial u}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right];
\end{aligned}$$

Since f is analytic, its real and imaginary parts satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0,$$

so the middle-line term on the right-hand side vanishes. Likewise, the top-line term equals

$$\begin{aligned}
&\frac{\partial^2 \phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial^2 \phi}{\partial v^2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] =\\
&= \frac{\partial^2 \phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2\right] + \frac{\partial^2 \phi}{\partial v^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] =\\
&= \left[\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\\
&= [\Delta \phi]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0,
\end{aligned}$$

since $\Delta\phi = 0$ for harmonic function ϕ.

Finally, the bottom-line terms on the right-hand side vanish because $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \Delta v = 0$, since the real and imaginary parts u, v of an analytic function $f = u + iv$ are harmonic functions. We conclude that $\Delta\psi = 0$. □

4. Find a Möbius transform mapping $0, 1, i$ to $\infty, 1, -i$, respectively. Is it unique?

Solution: Let $m(z) = (az + b)/(cz + d)$ and determine a, b, c, d satisfying the stated conditions and also $ad - bc = 1$. But then

$$\begin{aligned} m(0) = \infty &\Rightarrow d = 0; \\ m(1) = 1 &\Rightarrow a + b = c + d = c; \\ m(i) = -i &\Rightarrow ai + b = c - id = c. \end{aligned}$$

Together, these three conditions force $a = 0$, $b = c$, and $d = 0$. With $ad - bc = 1$ we get $b = c = \pm i$. That gives the Möbius transform $m(z) = 1/z$. It is unique by Exercise 25.2 and the discussion preceding it on page 193 of our textbook. □

5. Find all the Möbius transforms mapping the disk $\{|z| < 1\}$ to its exterior $\{|z| > 1\}$.

Solution: First note that $m(z) = 1/z$ is one Möbius transformation that maps $\{|z| < 1\}$ to $\{|z| > 1\}$. Then note that any such Möbius transformation may be written as the composition $m \circ n$, where n is a Möbius transform of the unit disk $\{|z| < 1\}$ into itself. But by Exercise 25.4 on page 197 of our textbook, there is a two-parameter family

$$n_{\alpha,\lambda}(z) \stackrel{\text{def}}{=} e^{i\lambda}\frac{z - \alpha}{\bar{\alpha}z - 1}; \qquad \lambda \in \mathbf{R}, |\alpha| < 1,$$

giving all the Möbius transforms of $\{|z| < 1\}$. (Note: our condition $ad - bc = 1$ is met by using the more complicated formula $(a, b, c, d) = k(e^{i\lambda}, -\alpha e^{i\lambda}, \bar{\alpha}, -1)$, where $k = \pm i e^{-i\lambda/2}/\sqrt{1 - |\alpha|^2}$.) Hence the complete set of Möbius transforms we seek is

$$\{M(z) = 1/n_{\alpha,\lambda}(z) = e^{i\lambda}\frac{\bar{\alpha}z - 1}{z - \alpha} : \lambda \in \mathbf{R}, |\alpha| < 1\}.$$

□