Ma 450: Mathematics for Multimedia
Final Examination

NAME:____________________________________

Due 4:00 pm Friday, May 2nd, 2014
in Cupples I, room 100 (Math Dept office)
9 problems on 9 pages

You may use a calculator or computer. You may consult any existing written or online reference material, but you may not get help from, or collaborate with, any other person. Please write your complete answers in the space provided.

1. Complete the following table, for all listed values of B, by finding:

   X, the string of digits in $\{0, 1, \ldots, B-1\}$ satisfying $X(\text{base } B) = 17(\text{base } 10)$, and

   Y, the number in $\{0, 1, \ldots, B-1\}$ satisfying $17 \equiv Y \pmod{B}$.

   NOTE: Use a, b, c, d, e, f, g for the digits $10, 11, 12, 13, 14, 15, 16$, respectively.

B	X (base B) = 17 (base 10)	$Y \equiv 17 \pmod{B}$
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17		

2. Let X be the inner product space of real-valued infinite sequences $x = (x_1, x_2, \ldots)$ with inner product

$$\langle x, y \rangle \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} 10^{-n} x_n y_n.$$

a. Compute the derived norm $\|x\|$ for the constant sequence $x = (1, 1, 1, \ldots)$.

b. Find an orthonormal basis for X.

c. Find $\|T\|_{\text{op}}$, where $T : X \to X$ is the right-shift operator, defined by

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

3. Let $x = (x_0, x_1, x_2, x_3) = (1, 1, -1, 1)$.
   a. Compute the discrete Hartley transform of the 4-point sequence x.
   b. Compute the discrete Fourier transform of the 2-point sequence x^{odd}.
   c. Compute the discrete Fourier transform of the 2-point sequence x^{even}.

4. Let $Z = \{(0,0), (2,4), (3,7), (4,8), (5,3)\}$ be an interpolation set.

   a. Use the divided differences method to find Newton's form of the Lagrange polynomial $p(x)$ through the points in Z.

   b. Evaluate $p(1)$.

5. The following function is sometimes called the *Shannon wavelet*:

$$w(x) \stackrel{\text{def}}{=} \frac{\sin(2\pi x) - \sin(\pi x)}{\pi x}.$$

a. Express w as a combination of sinc functions.

b. Show that w is band-limited and determine its bandwidth.

c. Show that w is admissible and compute its admissibility constant.

6. Let $f : \mathbf{Aff} \to \mathbf{R}$ be the characteristic function

$$f(a,b) = \begin{cases} 1, & \text{if } 0 < b < 4 \text{ and } 4 < a < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\int_{\mathbf{Aff}} f(a,b)\, d(a,b)$ using the normalized left-invariant integral.

7. Compute the complete discrete Haar wavelet transform, by the lifting method, on the signal u defined by $u(n) = 1$ for $n = 0, 1, \ldots, 7$.

8. Let $A = \{a, b, c, d, e\}$ be an alphabet with an information source that has occurrence probabilities $p = (.05, .10, .20, .25, .40)$.

   a. Construct a canonical Huffman code for A with the property that no letter has a codeword consisting of just 1-bits.

   b. Compute the bit rate for the code you constructed in part (a).

   c. Compute the theoretical minimum bit rate for the information source.

9. Let $p = p(t) = t^2 + t + 1$ and $q = q(t) = t^3 + t^2 + t + 1$ be mod-2 polynomials.

   a. Find the sum $p + q$.

   b. Find the product pq.

   c. Show that p is irreducible or find a factorization.

   d. Show that q is irreducible or find a factorization.

   e. Find the remainder $q\%p$.