Ma 450: Mathematics for Multimedia
Final Examination

NAME:______________________________________

Due 12:00 noon Friday, May 5th, 2017
in Cupples I, room 105a (my office)
8 problems on 1+8 pages

You may use a calculator or computer. You may consult any existing written or online reference material, but you may not get help from, or collaborate with, any other person. Please write your complete answers in the space provided.

1. Let N be your student ID number.

   a. Express N in base-16 notation.

   b. Find the prime factorization of N.

   c. Find the Euler totient $\phi(N)$.

   d. Find an integer $Y \in \{0, 1, 2, \ldots, N-1\}$ satisfying $Y \equiv 2017^{\phi(N)} \pmod{N}$, or prove that none exists.

2. Let X be the inner product space of bounded real-valued infinite sequences $x = (x_1, x_2, x_3, \ldots)$ with inner product

$$\langle x, y \rangle \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} 2^{-i} x_i y_i$$

   a. Compute the derived norm $\|x\|$ for the constant sequence $x = (1, 1, 1, \ldots)$.

   b. Find an orthonormal basis for X.

   c. Find $\|T\|_{\text{op}}$, where $T : X \to X$ is the left shift operator defined by

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

3. Let $x = (x_0, x_1, x_2, x_3) = (1, 1, -1, 1)$.
   a. Compute the discrete Fourier transform of the 4-point sequence x.
   b. Compute the discrete Fourier transform of the 2-point sequence x^{odd}.
   c. Compute the discrete Fourier transform of the 2-point sequence x^{even}.

4. The following function is sometimes called the *Mexican hat wavelet*:

$$w(x) \stackrel{\text{def}}{=} (1 - 2\pi x^2)\, e^{-\pi x^2},$$

and $g(x) \stackrel{\text{def}}{=} e^{-\pi x^2}$ is often called the *gaussian function.*

a. Compute the Fourier integral transform $\mathcal{F}(x^2 g(x))$ in terms of $\mathcal{F}g$. (Hint: differentiate under the integral, or equivalently integrate by parts.)

b. Find the Fourier integral transform $\mathcal{F}w$.

c. Show that w is admissible and compute its admissibility constant. (Hint: use Macsyma.)

5. Let $f = f(x, y)$ be a function defined by

$$f(x,y) \stackrel{\text{def}}{=} \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi}{2}(x-y)\right), & \text{if } 0 \le y \le 1 \text{ and } y-1 \le x \le y+1; \\ 0, & \text{otherwise.} \end{cases}$$

a. Compute $\iint_{\mathbf{R}^2} f(x, y)\, dxdy$ and show that f is a (joint) probability density function. (Hint: use Macsyma.)

b. Compute the normalizing constant c_y and determine $f(x|y)$.

c. Compute the expectation $E(x|y)$. Is $d(x) = x$ an unbiased estimator?

d. Compute the risk $R(d, y)$ for the decision function $d(x) = x$. Does it depend on y?

6. Let $\psi(t) = \phi(2t) - \phi(2t-1)$ be the Haar mother wavelet, where ϕ is the characteristic function of the interval $[0, 1)$. For integers j, k, put

$$\psi_{jk}(t) \stackrel{\text{def}}{=} 2^{-j/2}\psi(2^{-j}t - k),$$

so that $\{\psi_{jk} : j, k \in \mathbf{Z}\}$ is the Haar orthonormal basis for $L^2(\mathbf{R})$. Find the Haar expansion coefficients $\{c_{jk} : j, k \in \mathbf{Z}\}$ for the function $f(t) = \phi(t-1)$ (the characteristic function of $[1, 2)$), namely the numbers satisfying

$$f(t) = \sum_{j,k \in \mathbf{Z}} c_{jk}\psi_{jk}(t).$$

(Hint: graph some of the ψ_{jk}.)

7. Let $A = \{a, b, c, d, e\}$ be an alphabet with an information source that has occurrence probabilities $p = (.01, .02, .17, .30, .50)$.

   a. Construct a canonical Huffman code for A with the property that no letter has a codeword consisting of just 1-bits.

   b. Compute the bit rate for the code you constructed in part (a).

   c. Compute the theoretical minimum bit rate for the information source.

8. Let $p = p(t) = t^2 + t + 1$ and $q = q(t) = t^3 + t^2 + t + 1$ be mod-2 polynomials.

   a. Find the sum $p + q$.

   b. Find the product pq.

   c. Show that p is irreducible or find a factorization.

   d. Show that q is irreducible or find a factorization.

   e. Find the remainder $q\%p$.