# Ma 450: Mathematics for Multimedia
# Final Examination

NAME:____________________________________

*Due 11:00 PM Friday, May 5th, 2023*

9 problems on 1+9 pages

You may use a calculator or computer. You may consult any existing written or online reference material, but you may not get help from, or collaborate with, any other intelligence. Please upload your complete answers using GradeScope.

1. Let $T_n$ denote the $n^{\text{th}}$ Chebyshev polynomial.

   a. Determine with proof all elements of the set $\{T_n(\frac{\sqrt{2}}{2}) : n \in \mathbf{N}\}$.

   b. Determine with proof all elements of the set $\{n \in \mathbf{N} : T_n(0) = 1\}$.

   c. Find a formula, valid for all $n \geq 2$, giving $T_{n+1}(x)$ in terms of $T_{n-1}(x)$ and $T_{n-2}(x)$.

2. Let $N$ be your student ID number.

   a. Express $N$ in base-16 notation.

   b. Find the prime factorization of $N$.

   c. Find the Euler totient $\phi(N)$.

   d. Find an integer $Y \in \{0, 1, 2, \ldots, N-1\}$ satisfying $2023Y \equiv 1 \pmod{N}$, or prove that none exists.

3. Let $X$ be the vector space of bounded real-valued infinite sequences $x = (x_1, x_2, x_3, \ldots)$ with inner product

$$\langle x, y \rangle \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} 7^{-n} x_n y_n$$

a. Compute the derived norm $\|x\|$ for the constant sequence $x = (1, 1, 1, \ldots)$.

b. Find an orthonormal basis for $X$.

c. Find $\|T\|_{\text{op}}$, where $T : X \to X$ is the linear transformation defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, 0, x_6, \ldots),$$

namely $T$ returns the input sequence with its odd-indexed coefficients set to 0.

4. Let $x = (x_0, x_1, \ldots, x_{15})$, $x_n = \cos(\pi n/8)$, be the sequence generated by Octave/MATLAB commands `t=0:15; x=cos(pi*t/8);`.

   a. Compute the discrete Fourier transform of the 16-point sequence $x$.

   b. Compute the periodic discrete wavelet transform of $x$ using Daubechies' filters of length 4.

   (Hint: use `d4.m` and `pcqfilt.m` from the class website.)

5. Write $g(x) = e^{-\pi x^2}$ for the gaussian function. Its second derivative is sometimes called the l.o.g., or *Laplacian of the Gaussian*:

$$w(x) \stackrel{\text{def}}{=} \frac{d^2}{dx^2} e^{-\pi x^2}.$$

a. Find the formula for $w(x)$ by computing the derivatives.

b. Find the $L^2$ norm $\|w\|$.

c. Find the Fourier integral transform $\mathcal{F}w$.

d. Show that $w$ is admissible and compute its admissibility constant.

(Hint: use Macsyma.)

6. Let $f = f(x, y)$ be a function on $\mathbf{R}^2$ defined by

$$f(x,y) \stackrel{\text{def}}{=} \begin{cases} K, & \text{if } y - \frac{1}{2} < x < y + \frac{1}{2} \text{ and } 1 \le y \le 11; \\ 0, & \text{otherwise,} \end{cases}$$

where $K$ is a constant.

a. Determine the value of $K$ for which $f$ is a (joint) probability density function.

b. Compute the normalizing constant $c_y = \int_{\mathbf{R}} f(x, y)\, dx$ and determine $f(x|y)$.

c. Suppose that $y \in [1, 11]$. Compute the expectation $E(x|y)$. Is $d(x) = x$ an unbiased estimator for this set of $y$ values?

d. Suppose that $y \in [1, 11]$. Compute the risk $R(d, y)$ for the decision function $d(x) = x$. Does it depend on $y$?

7. Find the Haar wavelet expansion of the function $f(t) = \mathbf{1}_{[4,8)}$ (the characteristic function of $[4, 8)$), namely find the numbers $\{c_{jk} : j, k \in \mathbf{Z}\}$ satisfying

$$f(t) = \sum_{j,k\in\mathbf{Z}} c_{jk}\psi_{jk}(t),$$

where $\psi(t)$ is the Haar mother function and $\psi_{jk}(t) = 2^{-j/2}\psi(\frac{t-k}{2^j})$. (Hint: use Equation 5.4 from the textbook.)

8. Let $A = \{a, b, c, d, e, f, g\}$ be an alphabet with an information source that has occurrence probabilities $p = (.01, .02, .03, .04, .20, .30, .40)$.

   a. Construct a canonical Huffman code for $A$ with the property that no letter has a codeword consisting of just 1-bits.

   b. Compute the bit rate for the code you constructed in part (a).

   c. Compute the theoretical minimum bit rate for the information source.

9. Let $p = p(t) = t^3 + t + 1$ and $q = q(t) = t^4 + t^2 + t + 1$ be mod-2 polynomials.

   a. Find the sum $p + q$.

   b. Find the product $pq$.

   c. Show that $p$ is irreducible or find a factorization.

   d. Show that $q$ is irreducible or find a factorization.

   e. Find the remainder $q \% p$.