Ma 450: Mathematics for Multimedia
Homework Assignment 2

Prof. Wickerhauser

Due Sunday, February 19th, 2023

1. Let N be a fixed positive integer.
   (a) How many vertices are there in the unit cube in Euclidean N-space?
   (b) Fix a vertex in the N-cube. How may other vertices are connected to it by single edges?
   (c) Use parts a and b to count the total number of edges in the N-cube.
2. Let $\mathbf{P}, \mathbf{Q}, \mathbf{S}$ be subspaces of $\mathbf{R}^N$ with respective dimensions p, q, s. Suppose that $\mathbf{S} = \mathbf{P} + \mathbf{Q}$.
   (a) Prove that $\max\{p, q\} \leq s \leq p + q$.
   (b) Find an example that achieves the equality $s = \max\{p, q\}$.
   (c) Find an example that achieves the equality $s = p + q$.
3. Prove Inequality 2.15 for every N.
4. Prove that $\|\mathbf{x} - \mathbf{y}\| \geq \big|\|\mathbf{x}\| - \|\mathbf{y}\|\big|$ for any vectors $\mathbf{x}, \mathbf{y}$ in a normed vector space $\mathbf{X}$.
5. Suppose that $\mathbf{Y}$ is an m-dimensional subspace of an N-dimensional inner product space $\mathbf{X}$. Prove that $\mathbf{Y}^\perp$ is at most $N - m$ dimensional.
6. Suppose that $\mathbf{Y} = \operatorname{span}\{\mathbf{y}_n : \; n = 1, \ldots, N\}$ and $\mathbf{Z} = \operatorname{span}\{\mathbf{z}_m : \; m = 1, \ldots, M\}$ are subspaces in an inner product space $\mathbf{X}$. Show that if $\langle \mathbf{y}_n, \mathbf{z}_m \rangle = 0$ for all n, m, then $\mathbf{Y} \perp \mathbf{Z}$.
7. Find an orthonormal basis for the subspace of $\mathbf{E}^4$ spanned by the vectors $\mathbf{x} = (1, 0, 0, 0)$, $\mathbf{y} = (1, 0, 1, 0)$, and $\mathbf{z} = (1, 1, 1, 0)$.
8. Find the biorthogonal dual of the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbf{E}^3$.
9. Confirm, by checking the necessary properties, that the inner product on **Poly** given by
   $$\langle p, q \rangle \stackrel{\text{def}}{=} \sum_k \bar{a}_k b_k,$$
   is Hermitean symmetric, nondegenerate, and linear. Here $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $q(x) = b_0 + b_1 x + \cdots + b_m x^m$, and the sum is over all nonzero terms $\bar{a}_k b_k$. Note that this inner product defines the derived norm in Equation 2.21.
10. Show that $\|T\|_{\text{op}}$ is infinite for $T : \mathbf{Poly} \to \mathbf{Poly}$ defined by $Tp(x) = \frac{d}{dx} p(x)$ (the derivative), with respect to the norm in Equation 2.21.

11. Suppose that A is an $N \times N$ matrix satisfying $A^k = Id$ for some integer $k > 0$. Prove that $\|A\|_{\mathrm{HS}} \geq 1$.

12. Can there be matrices $A, B \in \mathbf{Mat}(N \times N)$ satisfying $AB - BA = Id$?

13. Determine whether the linear transformation $T : \ell^2 \to \ell^2$ defined by

$$T(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2 + x_3}{2}, \frac{x_4 + x_5 + x_6}{3}, \ldots)$$

is bounded or unbounded.

14. Given a matrix $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, find a Givens rotation G such that GA is upper triangular.

15. Suppose that A and B are $N \times N$ matrices satisfying the condition $A(i,j) = B(i,j) = 0$ if $i > j$. Prove that their product satisfies the same condition. (This shows that the product of upper-triangular matrices is upper-triangular.)