Ma 450: Mathematics for Multimedia
Homework Assignment 3

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Due Sunday, March 5th, 2023

1. Suppose that $f(t) = t^2$ if $-1 \le t < 1$, and $f(t) = 0$ for all $t \notin [-1, 1)$. Find $f_1(t)$, the 1-periodization of f.

2. Define the *reflection* R to be the transformation $Ru(t) \stackrel{\text{def}}{=} u(-t)$ acting on the vector space of functions of one real variable. Let F be the fraying operator of Equation 3.14.

   (a) Show that R is a linear transformation.

   (b) Find a formula for the compositions RF, FR, and RFR.

3. Show that the set of functions $\{\sqrt{2}\sin \pi nt : n = 1, 2, \ldots\}$ is orthonormal with respect to the inner product

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_0^1 f(t)g(t)\, dt.$$

   That is, show that

$$\left\langle \sqrt{2}\sin \pi nt, \sqrt{2}\sin \pi mt \right\rangle = \begin{cases} 0, & \text{if } n \ne m, \\ 1, & \text{if } n = m, \end{cases}$$

   for all $n, m \in \mathbf{Z}^+$.

4. Compute the sine-cosine Fourier series of the 1-periodic function $f(x) = \cos^2(5\pi x)$. (Hint: use a trigonometric identity.)

5. Show that if $|c(n)| < 2^{-|n|}$ for all integers $n \ne 0$, then the 1-periodic function $f = f(t)$ which is the inverse Fourier transform of the sequence $\{c(n)\}$ must have a continuous d^{th} derivative for every positive integer d.

6. Suppose that ϕ has Fourier integral transform $\mathcal{F}\phi$.

   (a) Fix $k \in \mathbf{R}$ and let $\phi_k(x) \stackrel{\text{def}}{=} \phi(x + k)$. Show that $\mathcal{F}\phi_k(\xi) = e^{2\pi i k\xi}\mathcal{F}\phi(\xi)$.

   (b) Fix $a > 0$ and let $\phi_a(x) \stackrel{\text{def}}{=} \phi(ax)$. Show that $\mathcal{F}\phi_a(\xi) = \frac{1}{a}\mathcal{F}\phi(\xi/a)$.

7. Compute the inverse Fourier integral transform of the function

$$\psi(\xi) = \begin{cases} 1, & \text{if } -2 \le \xi < -1 \text{ or } 1 < \xi \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

   (Hint: notice that $\psi(\xi) = \mathbf{1}_I(\xi/4) - \mathbf{1}_I(\xi/2)$, where $I = [-\frac{1}{2}, \frac{1}{2}]$.)

8. Compute the Fourier integral transform of the bump function

$$b(x) = \begin{cases} 2 - 2|x|, & \text{if } -1 \leq x < -\frac{1}{2} \text{ or } \frac{1}{2} < x \leq 1; \\ 1, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

9. Show that the vectors $\bar{\omega}_n \in \mathbf{C}^N$, $n = 0, 1, \ldots, N-1$ defined by $\bar{\omega}_n(k) = \exp\left(-2\pi i n k/N\right)$ form an orthonormal basis with respect to the inner product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \frac{1}{N} \sum_{k=0}^{N-1} \overline{f(k)}\, g(k).$$

10. Write out explicitly the matrices for the 3×3 discrete inverse Fourier and Hartley transforms (F_3^{-1} and H_3^{-1}).

11. What is the matrix $(C_N^{IV})^2$ of the square of $N \times N$ DCT-IV? Give a formula for every positive integer N.