Ma 450: Mathematics for Multimedia
Homework Assignment 4

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Due Sunday, April 2nd, 2023

1. Fix $h > 0$. Given y_-, y_+, let $p = p(x)$ be the Lagrange polynomial through the points $(-h, y_-)$, $(0, 0)$, and (h, y_+).

   (a) [6 points] Find a formula for the value $y = p(x)$ in terms of h, x, y_-, and y_+.

   (b) [4 points] Find $p''(0)$ from the formula in part (a).

2. [10 points] Let $f(x) = x^2 + 1$ for $x \in [-1, 1]$. Find the expansion coefficients c_0, c_1, c_2 for f in terms of Chebyshev polynomials $T_0(x), T_1(x), T_2(x)$, namely

$$f(x) = c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x).$$

3. Suppose $x_1 < x_2$, $y_1 < 0$, and $y_2 > 0$. Let f be the piecewise linear function interpolating the set $\{(x_1, y_1), (x_2, y_2)\}$.

   (a) [5 points] On what interval (if any) is $f > 0$?

   (b) [5 points] On what interval (if any) is $f < 0$?

4. Suppose that we have a machine that, given a random number N of pennies, wraps them into bundles of 50, keeping 0 to 49 leftover pennies as its commission, and gives back $b(N)$ wrapped bundles. Let $50 * b(N)$ be the estimate for the number of pennies N measured by this "instrument."

   (a) [5 points] What is the quantization error of this instrument?

   (b) [5 points] What is the imprecision?

   (c) [5 points] What is the inaccuracy?

   (d) [5 points] Is this instrument calibrated?

5. Let $f(x) = \cos(x) + 2\sin(x)$ for $0 \leq x \leq 10$. Note that $f \in L^2([0, 10])$.

   Let $x_k = k$ and $y_k = f(x_k)$ for $k = 0, 1, \ldots, 10$ be an interpolation set.

   Estimate the signal-to-noise ratio in decibels for the following sampling approximations s to f, using Octave and a grid of evaluation points in $[0, 10]$ with spacing 0.01:

   (a) [10 points] The piecewise constant approximation using sampling function $\mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}$.

   (b) [10 points] The piecewise linear approximation using the hat function.

   (c) [10 points] The cubic spline approximation (so s is the natural cubic spline defined by the interpolation set).

Hint: Compute $\|f\|^2$, $\|s\|^2$, and $\|f-s\|^2$ as sums of squares at the evaluation points $\{0, 0.01, 0.02, \ldots, 10\}$.

6. [10 points] Let $f(x) = \cos(x) + 2\sin(x)$ for $0 \leq x \leq 10$. Note that $f \in L^2([0, 10])$.

   Let s be the band-limited approximation to f with bandwidth 1, namely

$$s(x) = \sum_{n=0}^{10} f(n) \operatorname{sinc}(x - n).$$

   Estimate the signal-to-noise ratio in decibels for this approximation as in the previous problem,

   Hint: Octave has a built-in `sinc()`. Use it in your own function for s, then compute $\|f\|^2$, $\|s\|^2$, and $\|f - s\|^2$ as sums of squares at the evaluation points $\{0, 0.01, 0.02, \ldots, 10\}$.

7. Let $f = f(x, y)$ be the joint probability density supported on the region $R = \{(x, y) : 0 \leq x \leq 1, x-1 \leq y \leq x + 1\}$ and defined by the formula $f(x, y) = 1 - |y - x|$ for $(x, y) \in R$, with $f(x, y) = 0$ elsewhere.

   (a) [5 points] Show that $\iint_R f(x, y)\, dxdy = 1$.

   (b) [5 points] Compute the normalizing constant c_x and determine $f(y \,|\, x)$.

   (c) [5 points] Compute the expectation $E(y \,|\, x)$. Is $d(x) = x$ an unbiased estimator?

   (d) [5 points] Compute the risk $R(d, y)$ for the decision function $d(x) = x$. Does it depend on y?