Ma 450: Mathematics for Multimedia
Homework Assignment 5

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Due Sunday, April 16th, 2023

All solutions are worth 10 points.

1. Draw the graphs of $w(t)$, $w(t/2)$, and $w(3t)$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

2. Draw the graphs of $w(\frac{t}{3}-4)$ and $w(\frac{t-4}{3})$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

3. Let $f = f(\mathbf{a}) = f(a,b)$ be the function on $\mathbf{Aff}$ defined by $f(\mathbf{a}) = \mathbf{1}_D(\mathbf{a})$, where $\mathbf{1}_D$ is the indicator function of the region $D = \{\mathbf{a} = (a,b) : A < a < A', B < b < B'\} \subset \mathbf{Aff}$ for $0 < A < A'$ and $-\infty < B < B' < \infty$. Evaluate $\int_{\mathbf{Aff}} f(\mathbf{a})\, d\mathbf{a}$ using the normalized left-invariant integral on $\mathbf{Aff}$.

4. Let $w = w(t)$ be the Haar mother function and define

$$\phi^J_{M,K}(t) \stackrel{\text{def}}{=} \sum_{j=M+1}^{M+J} \frac{1}{2^j} w\left(\frac{t-K}{2^j}\right)$$

   for arbitrary fixed $K \in \mathbf{R}$ and $M, J \in \mathbf{Z}$ with $J > 0$.

   a. Show that
$$\lim_{J\to\infty} \phi^J_{M,K}(t) = 2^{-M}\mathbf{1}_{[K,K+2^M)}(t) \stackrel{\text{def}}{=} \phi_{M,K}(t),$$

   b. Show that $\left\langle \phi^J_{M,K}, u\right\rangle \to \langle \phi_{M,K}, u\rangle$ as $J \to \infty$ for any function $u \in L^2(\mathbf{R})$.

   (Hint: use Equation 5.4 and Lemma 5.1.)

5. Compute $\|w\|$, where
$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log \xi)^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \le 0. \end{cases}$$

   (Hint: use Plancherel's theorem and Equation B.6 in Appendix B.)

6. Let w be the function defined by

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log|\xi|)^2}, & \text{if } \xi \neq 0; \\ 0, & \text{if } \xi = 0. \end{cases}$$

Show that w is admissible and compute its normalization constant c_w.

7. Fix $A < 0$, $B > 0$, and $R > 1$ and suppose that $w = w(x)$ is a function satisfying $\mathcal{F}w(\xi) = 1$ if $RA < \xi < A$ or $B < \xi < RB$, with $\mathcal{F}w(\xi) = 0$, otherwise.

   a. Show that w satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant c_w.

   b. Give a formula for w.

8. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the antisymmetry condition $h(0) = -h(3)$ and $h(1) = -h(2)$, or prove that none exist.

9. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the symmetry condition $h(0) = h(3)$ and $h(1) = h(2)$, or prove that none exist.

10. Suppose that an orthogonal MRA has a scaling function ϕ satisfying $\phi(t) = 0$ for $t \notin [a, b]$. Prove that the low-pass filter h for this MRA must satisfy $h(n) = 0$ for all $n \notin [2a - b, 2b - a]$. (This makes explicit the finite support of h in Equation 5.36.)

11. Suppose that $h = \{h(k) : k \in \mathbf{Z}\}$ and $g = \{g(k) : k \in \mathbf{Z}\}$ satisfy the orthogonal CQF conditions. Show that the 2-periodizations h_2, g_2 of h and g are the Haar filters. Namely, show that $h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2}$.

12. Let ϕ be the scaling function of an orthogonal MRA, and let ψ be the associated mother function. For $(x, y) \in \mathbf{R}^2$, define

$$e_0(x, y) = \phi(x)\phi(y), \qquad e_1(x, y) = \phi(x)\psi(y)$$
$$e_2(x, y) = \psi(x)\phi(y), \qquad e_3(x, y) = \psi(x)\psi(y).$$

Prove that the functions $\{e_n : n = 0, 1, 2, 3\}$ are orthonormal in $L^2(\mathbf{R}^2)$, the inner product space of square-integrable functions on $\mathbf{R}^2$.