Ma 450: Mathematics for Multimedia
Homework Assignment 6

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Due Sunday, April 30th, 2023

1. Fix an integer $q \geq 2$, let $N = 2^q \geq 4$ and consider a graph with vertices labeled $0, 1, \ldots, N-1$. Suppose that vertex i is connected by an edge to vertex j if and only if the base-two expansions for i and j differ by exacly two bitflips. Compute the total number of edges.

2. Construct a prefix code for the alphabet A={a,b,c,d,e,f} with codeword lengths 1,3,3,3,4,4 or prove that none exists.

3. Construct a prefix code for the 24-letter Greek alphabet $A = \{\alpha, \beta, \gamma, \ldots, \omega\}$ with longest codeword 4, or prove that none exists.

4. Suppose we have two prefix codes, $\mathbf{c}_0(a,b) = (1,0)$ and $\mathbf{c}_1(a,b) = (0,1)$, for the alphabet $A = \{a, b\}$. Show that the following *dynamic encoding* is uniquely decipherable by finding a decoding algorithm:

   Simple Dynamic Encoding Example

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   dynamicencoding0( msg[], M ):
   [0] Initialize n=0
   [1] For m=1 to M, do [2] to [3]
   [2]    Transmit msg[m] using code n
   [3]    If msg[m]==‘b’, then toggle n = 1-n
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   (This encoding is called dynamic because the codeword for a letter might change as a message is encoded, in contrast with the *static encodings* studied in this chapter. It gives an example of a uniquely decipherable and instantaneous code which is nevertheless not a prefix code.)

5. A k-ary tree is called *extended* if every *interior*, or non-leaf, vertex has all k children. Let N_d be the number of extended k-ary trees of depth d or less.

   (a) Find, with proof, a recursive formula for N_{d+1} in terms of N_d.

   (b) Compute N_3.

6. Fix a positive integer n and consider the alphabet $A = \{a_1, \ldots, a_n, a_{n+1}\}$ with occurrence probabilities $p(a_i) = 2^{-i}$ for $i = 1, \ldots, n$, and $p(a_{n+1}) = 2^{-n}$.

   (a) Construct a Huffman code for the alphabet and compare its bit rate with $H(p)$.

   (b) Construct a canonical Huffman code for this alphabet, with the property that no letter has a codeword consisting of just 1-bits. Compute its bit rate.

7. (a) Find a binary code with five 13-bit or shorter codewords, wherein restoration to the nearest codeword corrects any three or fewer bit flips.

   (b) Can part (a) be solved with 12-bit codewords?

   (c) How many 13-bit codewords can you find satisfying the three-bitflip correction condition?

8. (a) Prove that casting out seventeens will detect all one-digit errors in hexadecimal arithmetic.

   (b) Find an example one-hexadecimal-digit error undetected by casting out fifteens.

9. Find a mod-2 polynomial of degree 4 that is relatively prime to $p(t) = t^7 + t^6 + t^3 + t$. (Hint: use Euclid's algorithm for mod-2 polynomials.)

10. Suppose that b is a prime number. Write $b = \dots b_2 b_1 b_0$ (base 2) and let $p(t) = b_0 + b_1 t + b_2 t^2 + \cdots$ be the associated mod-2 polynomial. Prove or find a counterexample to the claim that p must be irreducible.

11. Find integers j, k, $0 < k < j < 32$, such that $s(t) = t^{32} + t^j + t^k + 1$ is an irreducible mod-2 polynomial, or prove that none exists. (Hint: try dividing one such $s(t)$ by $t + 1$.)