Ma 450: Mathematics for Multimedia

Solution: to Homework Assignment 2

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Due Sunday, February 19th, 2023

1. Let N be a fixed positive integer.

   (a) How many vertices are there in the unit cube in Euclidean N-space?

   (b) Fix a vertex in the N-cube. How may other vertices are connected to it by single edges?

   (c) Use parts a and b to count the total number of edges in the N-cube.

   Solution: (a) There are 2^N vertices, representable by all distinct $\{0, 1\}$ sequences of length N.

   (b) Imagine aligning the N-cube with the coordinate axes in Euclidean N-space in such a way that the chosen vertex is at the origin and the edges at that vertex lie in the positive rays of the N coordinate axes. The vertices connected to the origin by single edges will then lie at coordinate 1 in each of the N directions, so there will be exactly N of them.

   (c) From part a, there are 2^N vertices. From part b, each of these 2^N vertices shares an edge with N others. Summing over the vertices counts each edge twice, so the number of edges is $N2^{N-1}$. □

2. Let $\mathbf{P}, \mathbf{Q}, \mathbf{S}$ be subspaces of $\mathbf{R}^N$ with respective dimensions p, q, s. Suppose that $\mathbf{S} = \mathbf{P} + \mathbf{Q}$.

   (a) Prove that $\max\{p, q\} \le s \le p + q$.

   (b) Find an example that achieves the equality $s = \max\{p, q\}$.

   (c) Find an example that achieves the equality $s = p + q$.

   Solution:

   (a) Let $\mathbf{P} = \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ and $\mathbf{Q} = \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$ define bases. Then $\mathbf{S} = \mathbf{P} + \mathbf{Q} = \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_p; \mathbf{v}_1, \dots, \mathbf{v}_q\}$, so by Theorem 2.2 the dimension of $\mathbf{S}$ is at most $p + q$.

   Vectors $\mathbf{u}_1, \dots, \mathbf{u}_p \subset P \subset S$ are linearly independent, so the dimension of $\mathbf{S}$ is at least p. Similarly, $s \ge q$. Thus $s \ge \max\{p, q\}$.

   (b) Use $\mathbf{R}^3$ with $\mathbf{P} = \operatorname{span}\{\mathbf{e}_1\}$ and $\mathbf{Q} = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$.

   (c) Use $\mathbf{R}^2$ with $\mathbf{P} = \operatorname{span}\{\mathbf{e}_1\}$ and $\mathbf{Q} = \operatorname{span}\{\mathbf{e}_2\}$. □

3. Prove Inequality 2.15 for every N.

   Solution: Fix N and let $\mathbf{x} \in \mathbf{C}^N$ be any vector. Then $|x(k)| \le \max\{|x(i)| : i = 1, \dots, N\} = \|\mathbf{x}\|_\infty$ for each $k = 1, \dots, N$, so

$$\|\mathbf{x}\|_1 = |x(1)| + \cdots + |x(N)| \le N\|\mathbf{x}\|_\infty.$$

Also, $|x(k)| \geq 0$ for all $k = 1, \dots, N$, so

$$\|\mathbf{x}\|_1 = |x(1)| + \cdots + |x(N)| \geq \max\{|x(i)| : i = 1, \dots, N\} = \|\mathbf{x}\|_\infty,$$

proving the other inequality.

Since $\mathbf{x}$ was arbitrary, the inequalites hold for all of $\mathbf{R}^N$. Since N was arbitrary, both equalities hold for any N. □

4. Prove that $\|\mathbf{x} - \mathbf{y}\| \geq \big| \|\mathbf{x}\| - \|\mathbf{y}\| \big|$ for any vectors $\mathbf{x}, \mathbf{y}$ in a normed vector space $\mathbf{X}$.

Solution: Use the norm sublinearity axiom twice:

$$\begin{aligned}
\|\mathbf{x}\| = \|(\mathbf{x} - \mathbf{y}) + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| \quad &\Rightarrow \quad \|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|; \\
\|\mathbf{y}\| = \|(\mathbf{y} - \mathbf{x}) + \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\| \quad &\Rightarrow \quad \|\mathbf{y} - \mathbf{x}\| \geq \|\mathbf{y}\| - \|\mathbf{x}\|.
\end{aligned}$$

But $\|\mathbf{y} - \mathbf{x}\| = \|(-1)(\mathbf{x} - \mathbf{y})\| = |-1|\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$, so the two inequalities may be combined:

$$\|\mathbf{x} - \mathbf{y}\| \geq \max\{\|\mathbf{y}\| - \|\mathbf{x}\|, \|\mathbf{x}\| - \|\mathbf{y}\|\} = \big| \|\mathbf{x}\| - \|\mathbf{y}\| \big|,$$

since $\max\{z, -z\} = |z|$ for any real number z. □

5. Suppose that $\mathbf{Y}$ is an m-dimensional subspace of an N-dimensional inner product space $\mathbf{X}$. Prove that $\mathbf{Y}^\perp$ is at most $N - m$ dimensional.

Solution: First check the trivial case: If $m = 0$, then $\mathbf{Y} = \{\mathbf{0}\}$, so $\mathbf{Y}^\perp = \mathbf{X}$ is $N = N - 0 = N - m$ dimensional.

Otherwise, let $\mathbf{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a basis for $\mathbf{Y} = \operatorname{span} \mathbf{V}$. Since $\mathbf{Y}^\perp$ is a subspace of a finite-dimensional space, it too is finite-dimensional, so let $\mathbf{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be its basis. The dimension of $\mathbf{Y}^\perp$ is k, so it remains to show that $m + k \leq N$.

Now suppose $a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m + b_1\mathbf{w}_1 + \cdots + b_k\mathbf{w}_k = \mathbf{0}$. Then $a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = -(b_1\mathbf{w}_1 + \cdots + b_k\mathbf{w}_k)$ belongs to both $\mathbf{Y} = \operatorname{span} \mathbf{V}$ and $\mathbf{Y}^\perp = \operatorname{span} \mathbf{W}$. But $\mathbf{Y} \cap \mathbf{Y}^\perp = \{\mathbf{0}\}$ by Lemma 2.5, so this implies $a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = \mathbf{0}$ and $b_1\mathbf{w}_1 + \cdots + b_k\mathbf{w}_k = \mathbf{0}$. Thus $a_1 = \cdots = a_m = 0$ and $b_1 = \cdots = b_k = 0$ by the linear independence of sets $\mathbf{V}$ and $\mathbf{W}$ individually. This shows that $\mathbf{V} \cup \mathbf{W}$ is a linearly independent set in $\mathbf{X}$. There cannot be more than N linearly independent vectors in an N-dimensional vector space, so it follows that $m + k \leq N$. □

6. Suppose that $\mathbf{Y} = \operatorname{span}\{\mathbf{y}_n : n = 1, \dots, N\}$ and $\mathbf{Z} = \operatorname{span}\{\mathbf{z}_m : m = 1, \dots, M\}$ are subspaces in an inner product space $\mathbf{X}$. Show that if $\langle \mathbf{y}_n, \mathbf{z}_m \rangle = 0$ for all n, m, then $\mathbf{Y} \perp \mathbf{Z}$.

Solution: Let $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{z} \in \mathbf{Z}$ be arbitrary and write $\mathbf{y} = \sum_{n=1}^N a_n \mathbf{y}_n$ and $\mathbf{z} = \sum_{m=1}^M b_m \mathbf{z}_m$ for appropriate scalars $a_1, \dots, a_N$ and $b_1, \dots, b_M$. But then

$$\begin{aligned}
\langle \mathbf{y}, \mathbf{z} \rangle &= \left\langle \sum_{n=1}^N a_n \mathbf{y}_n, \sum_{m=1}^M b_m \mathbf{z}_m \right\rangle \\
&= \sum_{n=1}^N \sum_{m=1}^M a_n b_m \langle \mathbf{y}_n, \mathbf{z}_m \rangle \qquad = \quad 0,
\end{aligned}$$

since every term in the sum is zero by hypothesis. □

7. Find an orthonormal basis for the subspace of $\mathbf{E}^4$ spanned by the vectors $\mathbf{x} = (1,0,0,0)$, $\mathbf{y} = (1,0,1,0)$, and $\mathbf{z} = (1,1,1,0)$.

Solution: First note that these three vectors are linearly independent: if $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$, then $(a+b+c, c, b+c, 0) = (0,0,0,0)$, which implies $a = b = c = 0$. Applying the recursive Gram-Schmidt construction gives the orthonormal set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$, where

$$\begin{aligned} \mathbf{p} &= \frac{1}{\|\mathbf{x}\|}\mathbf{x} = (1,0,0,0)\,; \\ \mathbf{q}' &= \mathbf{y} - \langle \mathbf{p}, \mathbf{y}\rangle\, \mathbf{p} = (0,0,1,0) = \mathbf{q}; \\ \mathbf{r}' &= \mathbf{z} - \langle \mathbf{p}, \mathbf{z}\rangle\, \mathbf{p} - \langle \mathbf{q}, \mathbf{z}\rangle\, \mathbf{q} = (0,1,0,0) = \mathbf{r}, \end{aligned}$$

since both $\mathbf{q}'$ and $\mathbf{r}'$ happen to be unit vectors. □

8. Find the biorthogonal dual of the basis $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$ of $\mathbf{E}^3$.

Solution: This may be solved by undetermined coefficients. Let (a,b,c), (d,e,f), and (g,h,i) be the coordinates of the biorthogonal dual vectors. The biorthogonality conditions imply that $a = 1$, $a + b = 0$, $a + b + c = 0$; $d = 0$, $d + e = 1$, $d + e + f = 0$; $g = 0$, $g + h = 0$, $g + h + i = 1$. Back substitution gives $b = -1$, $c = 0$; $e = 1$, $f = -1$; $h = 0$, $i = 1$. Hence the dual vectors are $(1,-1,0)$, $(0,1,-1)$, and $(0,0,1)$.

Alternatively, use matrix inversion:

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Bt=[1,0,0; 1,1,0; 1,1,1]; inv(Bt)  % Transpose Bt has the basis as its rows
ans =
   1    0    0
  -1    1    0
  -0   -1    1
```

The biorthogonal dual basis is found in the columns. Note that -0 equals 0. □

9. Confirm, by checking the necessary properties, that the inner product on **Poly** given by

$$\langle p, q \rangle \stackrel{\text{def}}{=} \sum_k \bar{a}_k b_k,$$

is Hermitean symmetric, nondegenerate, and linear. Here $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $q(x) = b_0 + b_1 x + \cdots + b_m x^m$, and the sum is over all nonzero terms $\bar{a}_k b_k$. Note that this inner product defines the derived norm in Equation 2.21.

Solution: Hermitean symmetry follows from the identity

$$\overline{\left(\sum_k \bar{a}_k b_k \right)} = \sum_k a_k \bar{b}_k.$$

Nondegeneracy holds because for $p(x) = a_0 + a_1 x + \cdots + a_n x^n$,

$$\langle p, p \rangle = 0 \;\Rightarrow\; \sum_k |a_k|^2 = 0 \;\Rightarrow\; (\forall k = 0, 1, \ldots, n)\, a_k = 0.$$

But this means that p is the zero polynomial.

Linearity holds because for polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$, $q(x) = b_0 + b_1x + \cdots + b_mx^m$, $r(x) = c_0 + c_1x + \cdots + c_lx^l$, and scalars s, t,

$$\langle p, sq + tr\rangle = \sum_k \bar{a}_k(sb_k + tc_k) = s\sum_k (\bar{a}_kb_k) + t\sum_k (\bar{a}_kc_k).$$

This is evidently $s\langle p, q\rangle + t\langle p, r\rangle$. □

10. Show that $\|T\|_{\mathrm{op}}$ is infinite for $T : \mathbf{Poly} \to \mathbf{Poly}$ defined by $Tp(x) = \frac{d}{dx}p(x)$ (the derivative), with respect to the norm in Equation 2.21.

    Solution: Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $Tp(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$, so

    $$\|Tp\|^2 = |a_1|^2 + 2^2|a_2|^2 + \cdots + n^2|a_n|^2,$$

    whereas

    $$\|p\|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2.$$

    The ratio $\|Tp\|^2/\|p\|^2 = n$ for $p(x) = x^n$, and this is unbounded over **Poly**, which contains polynomials of arbitrarily large degree. Hence $\|T\|_{\mathrm{op}} = \infty$. □

11. Suppose that A is an $N \times N$ matrix satisfying $A^k = Id$ for some integer $k > 0$. Prove that $\|A\|_{\mathrm{HS}} \geq 1$.

    Solution: A direct calculation with Equation 2.44 shows that $\|Id\|_{\mathrm{HS}} = \sqrt{N} \geq 1$. Thus for some $k > 0$,

    $$1 \leq \|Id\|_{\mathrm{HS}} = \|A^k\|_{\mathrm{HS}} \leq \|A\|_{\mathrm{HS}}^k,$$

    by the submultiplicativity of the Hilbert-Schmidt norm. Taking k^{th} roots on both sides gives $1 \leq \|A\|_{\mathrm{HS}}$. □

12. Can there be matrices $A, B \in \mathbf{Mat}(N \times N)$ satisfying $AB - BA = Id$?

    Solution: No. Compare traces: $\mathrm{tr}\,(AB - BA) = 0$ by Equation 2.46, while $\mathrm{tr}\,(Id) = N \neq 0$. □

13. Determine whether the linear transformation $T : \ell^2 \to \ell^2$ defined by

    $$T(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2 + x_3}{2}, \frac{x_4 + x_5 + x_6}{3}, \ldots)$$

    is bounded or unbounded.

    Solution: T is bounded. First note that for every integer $k > 0$ and all scalars $x_{p+1}, x_{p+2}, \ldots, x_{p+k}$,

    $$\begin{aligned}\left(\frac{x_{p+1} + \cdots + x_{p+k}}{k}\right)^2 &= \frac{1}{k^2}\left|\langle(1, \ldots, 1), (x_{p+1}, \ldots, x_{p+k})\rangle\right|^2 \\ &\leq \frac{1}{k^2}\|(1, \ldots, 1)\|^2\left(|x_{p+1}|^2 + \cdots + |x_{p+k}|^2\right) \\ &= \frac{1}{k}\left(|x_{p+1}|^2 + \cdots + |x_{p+k}|^2\right),\end{aligned}$$

    using the Cauchy-Schwarz inequality in $\mathbf{E}^k$. Thus

    $$\|T(x_1, x_2, \ldots)\|^2 \leq |x_1|^2 + \frac{1}{2}(|x_2|^2 + |x_3|^2) + \frac{1}{3}(|x_4|^2 + |x_5|^2 + |x_6|^2) + \cdots \leq \|x\|^2.$$

    Conclude that T is bounded with $\|T\|_{\mathrm{op}} \leq 1$.

    Checking $x = (1, 0, 0, \ldots)$, we see that $\|Tx\| = 1 = \|x\|$, so in fact $\|T\|_{\mathrm{op}} = 1$. □

14. Given a matrix $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, find a Givens rotation G such that GA is upper triangular.

Solution: Write

$$G = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

with θ to be determined in terms of x, y, z, w. If GA is upper triangular, then its $2,1$ coefficient must be zero:

$$(-\sin\theta)x + (\cos\theta)z = 0.$$

If $z = 0$, then A is already upper triangular and it suffices to choose $\theta = 0$, making $G = Id$ the trivial Givens rotation.

If $z \neq 0$, then $x/z = \cos\theta/\sin\theta = \cot\theta$ will make GA upper triangular, so it suffices to choose $\theta = \cot^{-1}(x/z)$. □

15. Suppose that A and B are $N \times N$ matrices satisfying the condition $A(i,j) = B(i,j) = 0$ if $i > j$. Prove that their product satisfies the same condition. (This shows that the product of upper-triangular matrices is upper-triangular.)

Solution: Their product $C = AB$ is $C(i,j) = \sum_{k=1}^{N} A(i,k)B(k,j)$. But $A(i,k)B(k,j) = 0$ if $i > k$ or $k > j$, and one of these will be true for all $k \in \{1, \ldots, N\}$ if $i > j$. Thus $C(i,j) = 0$ for $i > j$. □