Ma 450: Mathematics for Multimedia

Solution: to Homework Assignment 3

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Due Sunday, March 5th, 2023

1. Suppose that $f(t) = t^2$ if $-1 \le t < 1$, and $f(t) = 0$ for all $t \notin [-1, 1)$. Find $f_1(t)$, the 1-periodization of f.

   Solution: Since f_1 is 1-periodic, it suffices to find its values on one period interval, such as $[0, 1)$. Since $f(t) = 0$ unless $t \in [-1, 1)$, the sum defining $f_1(t) = \sum_{k\in\mathbf{Z}} f(t + k)$ has exactly two nonzero terms. With the assumption that $t \in [0, 1)$, these are $f(t)$ and $f(t - 1)$. But then

$$f_1(t) = f(t) + f(t-1) = t^2 + (t-1)^2 = 2t^2 - 2t + 1.$$

□

2. Define the *reflection* R to be the transformation $Ru(t) \stackrel{\text{def}}{=} u(-t)$ acting on the vector space of functions of one real variable. Let F be the fraying operator of Equation 3.14.

   (a) Show that R is a linear transformation.

   (b) Find a formula for the compositions RF, FR, and RFR.

   Solution: (a) R is a linear transformation, since $R[au+bv](t) = au(-t)+bv(-t) = aRu(t)+bRv(t)$. The compositions RF, FR, and RFR are therefore linear transformations as well.

   (b) Using Equation 3.14 gives the following formulas:

$$\begin{aligned}
RFu(t) &= \begin{cases} r(-t)u(-t) + r(t)u(t), & \text{if } -t > 0, \\ \bar{r}(t)u(-t) - \bar{r}(-t)u(t), & \text{if } -t < 0, \end{cases} \\
&= \begin{cases} \bar{r}(t)u(-t) - \bar{r}(-t)u(t), & \text{if } t > 0, \\ r(-t)u(-t) + r(t)u(t), & \text{if } t < 0; \end{cases} \\
FRu(t) &= \begin{cases} r(t)Ru(t) + r(-t)Ru(-t), & \text{if } t > 0, \\ \bar{r}(-t)Ru(t) - \bar{r}(t)Ru(-t), & \text{if } t < 0, \end{cases} \\
&= \begin{cases} r(t)u(-t) + r(-t)u(t), & \text{if } t > 0, \\ \bar{r}(-t)u(-t) - \bar{r}(t)u(t), & \text{if } t < 0; \end{cases} \\
RFRu(t) &= \begin{cases} r(-t)Ru(-t) + r(t)Ru(t), & \text{if } -t > 0, \\ \bar{r}(t)Ru(-t) - \bar{r}(-t)Ru(t), & \text{if } -t < 0, \end{cases} \\
&= \begin{cases} \bar{r}(t)u(t) - \bar{r}(-t)u(-t), & \text{if } t > 0, \\ r(-t)u(t) + r(t)u(-t), & \text{if } t < 0; \end{cases}
\end{aligned}$$

   In all cases, $RFu(0) = FRu(0) = RFRu(0) = u(0)$. □

3. Show that the set of functions $\{\sqrt{2}\sin \pi nt : n = 1, 2, \ldots\}$ is orthonormal with respect to the inner product

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_0^1 f(t)g(t)\,dt.$$

That is, show that

$$\left\langle \sqrt{2}\sin \pi nt, \sqrt{2}\sin \pi mt \right\rangle = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m, \end{cases}$$

for all $n, m \in \mathbf{Z}^+$.

Solution: First note that if k is a nonzero integer, then

$$\int_0^1 \cos \pi kt\,dt = \frac{1}{k\pi}[\sin k\pi - \sin 0] = 0.$$

Then, recall the angle addition formulas for cosine: for any real numbers A, B,

$$\begin{aligned} \cos(A - B) &= \cos A \cos B + \sin A \sin B, \\ \cos(A + B) &= \cos A \cos B - \sin A \sin B, \end{aligned}$$

so $2\sin A \sin B = \cos(A - B) - \cos(A + B)$. Thus the $n \neq m$ case yields

$$\left\langle \sqrt{2}\sin \pi nt, \sqrt{2}\sin \pi mt \right\rangle = 2\int_0^1 [\cos \pi(n-m)t - \cos \pi(n+m)t]\ dt = 0,$$

since both $n + m$ and $n - m$ are nonzero integers.

Finally, use the double angle formula $\cos 2A = 1 - 2\sin^2 A$ to compute

$$\left\langle \sqrt{2}\sin \pi nt, \sqrt{2}\sin \pi nt \right\rangle = 2\int_0^1 \sin^2 \pi nt\,dt = \int_0^1 [1 - \cos 2\pi nt]\ dt = 1,$$

which gives the $n = m$ case. □

4. Compute the sine-cosine Fourier series of the 1-periodic function $f(x) = \cos^2(5\pi x)$. (Hint: use a trigonometric identity.)

Solution: Since $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$, we have

$$\cos^2(5\pi x) = \frac{1}{2} + \frac{1}{2}\cos(5 \times 2\pi x).$$

Hence, the Fourier series

$$f(x) = a(0) + \sqrt{2}\sum_{n=1}^{\infty}[a(n)\cos(2\pi nx) + b(n)\sin(2\pi nx)]$$

has just two nonzero terms: $a(0) = \frac{1}{2}$ and $a(5) = \frac{1}{2\sqrt{2}}$, with $a(n) = 0$ for all $n \notin \{0, 5\}$ and $b(n) = 0$ for all n. □

5. Show that if $|c(n)| < 2^{-|n|}$ for all integers $n \neq 0$, then the 1-periodic function $f = f(t)$ which is the inverse Fourier transform of the sequence $\{c(n)\}$ must have a continuous d^{th} derivative for every positive integer d.

Solution: This is a straightforward application of Corollary 3.10. Fix a positive integer d and note that for any positive integer N,

$$\sum_{k=-N}^{N} |k|^{d+1}|c(k)| < 2\sum_{k=1}^{N} k^{d+1}2^{-k} < 2\sum_{k=1}^{\infty} k^{d+1}2^{-k},$$

and the infinite series converges by the ratio test, using the computation

$$\lim_{k\to\infty} \frac{(k+1)^{d+1}2^{-(k+1)}}{k^{d+1}2^{-k}} = \frac{1}{2}\left(\lim_{k\to\infty} \frac{k+1}{k}\right)^{d+1} = \frac{1}{2} < 1.$$

Thus $f = \check{c} \in \mathbf{Lip}$ and also $f^{(d)} \in \mathbf{Lip}$, so f has d continuous derivatives. □

6. Suppose that ϕ has Fourier integral transform $\mathcal{F}\phi$.

(a) Fix $k \in \mathbf{R}$ and let $\phi_k(x) \stackrel{\text{def}}{=} \phi(x+k)$. Show that $\mathcal{F}\phi_k(\xi) = e^{2\pi ik\xi}\mathcal{F}\phi(\xi)$.

(b) Fix $a > 0$ and let $\phi_a(x) \stackrel{\text{def}}{=} \phi(ax)$. Show that $\mathcal{F}\phi_a(\xi) = \frac{1}{a}\mathcal{F}\phi(\xi/a)$.

Solution: (a) Make the change of variable $x \leftarrow y - k$, so $dx \leftarrow dy$, and extract the factor $e^{2\pi ik\xi}$:

$$\begin{aligned}\mathcal{F}\phi_k(\xi) &= \int_{-\infty}^{\infty} e^{-2\pi ix\xi}\,\phi(x+k)\,dx \\ &= e^{2\pi ik\xi}\int_{-\infty}^{\infty} e^{-2\pi iy\xi}\,\phi(y)\,dy \quad = \quad e^{2\pi ik\xi}\mathcal{F}\phi(\xi).\end{aligned}$$

(b) Make the change of variable $x \leftarrow y/a$, so $dx \leftarrow \frac{1}{a}dy$, and extract the factor $1/a$:

$$\mathcal{F}\phi_a(\xi) = \int_{-\infty}^{\infty} e^{-2\pi ix\xi}\,\phi(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-2\pi iy(\xi/a)}\,\phi(y)\,dy = \frac{1}{a}\mathcal{F}\phi(\xi/a).$$

□

7. Compute the inverse Fourier integral transform of the function

$$\psi(\xi) = \begin{cases} 1, & \text{if } -2 \le \xi < -1 \text{ or } 1 < \xi \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

(Hint: notice that $\psi(\xi) = \mathbf{1}_I(\xi/4) - \mathbf{1}_I(\xi/2)$, where $I = [-\frac{1}{2}, \frac{1}{2}]$.)

Solution: From Exercise 12, with $f_a(x) \stackrel{\text{def}}{=} f(x/a)$, we get the identity

$$\mathcal{F}^{-1}f_a(\xi) = \mathcal{F}f_a(-\xi) = a\mathcal{F}f(-a\xi) = a\mathcal{F}^{-1}f(a\xi).$$

Thus, using the hint with $a = 2$ or $a = 4$ and the definition $\mathcal{F}\mathbf{1}_I = \operatorname{sinc}$, we calculate

$$\mathcal{F}^{-1}\psi(x) = 4\mathcal{F}\mathbf{1}_I(4x) - 2\mathcal{F}\mathbf{1}_I(2x) = 4\operatorname{sinc}(4x) - 2\operatorname{sinc}(2x).$$

□

8. Compute the Fourier integral transform of the bump function

$$b(x) = \begin{cases} 2-2|x|, & \text{if } -1 \le x < -\frac{1}{2} \text{ or } \frac{1}{2} < x \le 1; \\ 1, & \text{if } -\frac{1}{2} \le x \le \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Solution: First note that $b(x) = 2h(x) - h(2x)$, where h is the hat function of Exercise 14 on p.105 in the textbook:

$$h(x) = \begin{cases} 1-|x|, & \text{if } -1 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

But then $\mathcal{F}b(\xi) = 2\mathcal{F}h(\xi) - \frac{1}{2}\mathcal{F}h(\xi/2) = 2\text{sinc}\,(\xi)^2 - \frac{1}{2}\left[\text{sinc}\,(\xi/2)\right]^2$, using the results of Exercises 12 and 14 in section 3.3 of the textbook. □

9. Show that the vectors $\bar{\omega}_n \in \mathbf{C}^N$, $n = 0, 1, \ldots, N-1$ defined by $\bar{\omega}_n(k) = \exp(-2\pi i n k/N)$ form an orthonormal basis with respect to the inner product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \frac{1}{N} \sum_{k=0}^{N-1} \overline{f(k)}\, g(k).$$

Solution: Use the geometric series summation formula:

$$\begin{aligned} \frac{1}{N}\sum_{k=0}^{N-1} \overline{\bar{\omega}_n}(k)\bar{\omega}_m(k) &= \frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i n k}{N}} e^{-\frac{2\pi i m k}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i (n-m)k}{N}} \\ &= \begin{cases} \frac{1}{N}\frac{1-e^{2\pi i(n-m)}}{1-e^{2\pi i \frac{n-m}{N}}}, & \text{if } n \ne m; \\ \frac{1}{N}N, & \text{if } n = m, \end{cases} = \delta(n-m). \end{aligned}$$

Orthonormality implies linear independence, and any N linearly independent vectors in an N-dimensional vector space must be a basis. □

10. Write out explicitly the matrices for the 3×3 discrete inverse Fourier and Hartley transforms (F_3^{-1} and H_3^{-1}).

Solution: The normalized formulas $F_N^{-1}(m,n) = \frac{1}{\sqrt{N}}\exp(2\pi i m n/N)$ and $H_N^{-1}(m,n) = H_N(m,n) = \frac{1}{\sqrt{N}}\left[\cos(2\pi m n/N) + \sin(2\pi m n/N)\right]$ give:

$$F_3^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} \\ 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} \end{pmatrix};$$

$$H_3^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{-1+\sqrt{3}}{2} & \frac{-1-\sqrt{3}}{2} \\ 1 & \frac{-1-\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} \end{pmatrix}.$$

□

11. What is the matrix $(C_N^{IV})^2$ of the square of $N \times N$ DCT-IV? Give a formula for every positive integer N.

Solution: Since $\sqrt{\frac{2}{N}}\, C_N^{IV}$ is a symmetric unitary transformation, it is its own inverse:

$$\left(\sqrt{\frac{2}{N}}\, C_N^{IV}\right)^{-1} = \left(\sqrt{\frac{2}{N}}\, C_N^{IV}\right)^{*} = \sqrt{\frac{2}{N}}\, C_N^{IV}$$
$$\Rightarrow \left(\sqrt{\frac{2}{N}}\, C_N^{IV}\right)^{2} = Id \quad \Rightarrow \left(C_N^{IV}\right)^2 = \frac{N}{2} Id.$$

This is a direct consequence of Theorem 3.15. □