Ma 450: Mathematics for Multimedia

Solution: to Homework Assignment 4

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Due Sunday, April 2nd, 2023

1. Fix $h > 0$. Given y_-, y_+, let $p = p(x)$ be the Lagrange polynomial through the points $(-h, y_-)$, $(0, 0)$, and (h, y_+).

   (a) [6 points] Find a formula for the value $y = p(x)$ in terms of h, x, y_-, and y_+.

   (b) [4 points] Find $p''(0)$ from the formula in part (a).

   Solution:

   (a) Any method will work since the Lagrange polynomial is unique. It is most easily done with undetermined coefficients, using $p(x) = ax^2 + bx + c$ and noting that $p(0) = 0$ implies that $c = 0$:

   $$y_- = p(-h) = ah^2 - bh, \quad y_+ = p(h) = ah^2 + bh,$$

   so

   $$b = \frac{y_+ - y_-}{2h}, \quad a = \frac{y_+ + y_-}{2h^2}.$$

   The Lagrange polynomial is therefore

   $$p(x) = \frac{y_+ + y_-}{2h^2} x^2 + \frac{y_+ - y_-}{2h} x.$$

   (b) The second derivative of p with respect to x is in fact independent of x, so in particular its value at $x = 0$ is

   $$\frac{d^2}{dx^2} p(x) = \frac{y_+ + y_-}{h^2}.$$

   □

2. [10 points] Let $f(x) = x^2 + 1$ for $x \in [-1, 1]$. Find the expansion coefficients c_0, c_1, c_2 for f in terms of Chebyshev polynomials $T_0(x), T_1(x), T_2(x)$, namely

   $$f(x) = c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x).$$

   Solution: Since f itself is a polynomial of degree 2, it equals its Chebyshev polynomial expansion: $f(x) = c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x)$, for all $x \in [-1, 1]$. The expansion coefficients may thus be found by the method of undetermined coefficients. But $T_0(x) = 1$, $T_1(x) = x$, and $T_2(x) = 2x^2 - 1$, so:

   $$x^2 + 1 = c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot (2x^2 - 1) = [2c_2]x^2 + [c_1]x + [c_0 - c_2],$$

   so $c_2 = \frac{1}{2}$, $c_1 = 0$, and $c_0 = \frac{3}{2}$. □

3. Suppose $x_1 < x_2$, $y_1 < 0$, and $y_2 > 0$. Let f be the piecewise linear function interpolating the set $\{(x_1, y_1), (x_2, y_2)\}$.

   (a) [5 points] On what interval (if any) is $f > 0$?

   (b) [5 points] On what interval (if any) is $f < 0$?

   Solution: First use the point-slope formula through the points $(x_1, y_1), (x_2, y_2)$ to find $f(x) = m[x - x_1] + y_1$ by computing $m = (y_2 - y_1)/(x_2 - x_1)$. Note that $m > 0$ since both $y_2 - y_1 > 0$ and $x_2 - x_1 > 0$. Thus f is strictly increasing.

   Use Equation 4.14 to find $x_0 = \frac{x_2 y_1 - x_1 y_2}{y_1 - y_2}$, the root of the linear function $f = f(x)$. Since f is strictly increasing, it must satisfy:

   (a) $f(t) < 0$ for $t \in (-\infty, x_0)$, and

   (b) $f(t) > 0$ for $t \in (x_0, \infty)$. □

4. Suppose that we have a machine that, given a random number N of pennies, wraps them into bundles of 50, keeping 0 to 49 leftover pennies as its commission, and gives back $b(N)$ wrapped bundles. Let $50 * b(N)$ be the estimate for the number of pennies N measured by this "instrument."

   (a) [5 points] What is the quantization error of this instrument?

   (b) [5 points] What is the imprecision?

   (c) [5 points] What is the inaccuracy?

   (d) [5 points] Is this instrument calibrated?

   Solution: (a) The quantization error is half the 50-penny difference between possible reported values, or 25 pennies.

   (b) The imprecision is the root mean square error of a uniform density on the interval $[0, 49]$ with mean value 24.5:

$$\sqrt{\frac{1}{50}\sum_{p=0}^{49}(p - 24.5)^2} = \sqrt{\frac{2}{50}\sum_{p=0}^{24}(p - 24.5)^2},$$

   This may be evaluated with the Octave command

```
sqrt(var( 0:49, 1))
```

   which gives the imprecision as $\sqrt{208.25} \approx 14.43$ pennies. (The second argument in `var(0:49, 1)` specifies the option that the sum of N squares should be divided by N, not $N - 1$ as in the sample variance.)

   (c) The inaccuracy is the root mean square error between a uniform random variable taking integer values in the interval $[0, 49]$, and the measured value 0:

$$\sqrt{\frac{1}{50}\sum_{p=0}^{49}(p - 0)^2} = \sqrt{\frac{1}{50}\sum_{p=1}^{49}p^2}.$$

   But the sum of the squares of the integers $0, 1, 2, \ldots, N - 1$ is $\frac{1}{6}N(N+1)(2N+1)$ by Equation 0.121(2) on page 2 in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products*, fifth edition (1994), Academic Press, San Diego; ISBN 0-12-294755-X. This is 40425 for $N = 50$. Hence the inaccuracy is $\sqrt{40425/50} = \sqrt{808.5} \approx 28.43$ pennies.

(d) Since the reported count is never more than the number of pennies but may be up to 49 pennies too low, the expected ideal value is 24.5 pennies more than the reported value. Thus the counter is not calibrated. □

5. Let $f(x) = \cos(x) + 2\sin(x)$ for $0 \le x \le 10$. Note that $f \in L^2([0, 10])$.

   Let $x_k = k$ and $y_k = f(x_k)$ for $k = 0, 1, \ldots, 10$ be an interpolation set.

   Estimate the signal-to-noise ratio in decibels for the following sampling approximations s to f, using Octave and a grid of evaluation points in $[0, 10]$ with spacing 0.01:

   (a) [10 points] The piecewise constant approximation using sampling function $\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$.

   (b) [10 points] The piecewise linear approximation using the hat function.

   (c) [10 points] The cubic spline approximation (so s is the natural cubic spline defined by the interpolation set).

   Hint: Compute $\|f\|^2$, $\|s\|^2$, and $\|f-s\|^2$ as sums of squares at the evaluation points $\{0, 0.01, 0.02, \ldots, 10\}$.

Solution: Octave commands are appropriate since an approximation suffices. These are modified from the example file `https://www.math.wustl.edu/~victor/classes/ma450/octave.txt`.

```
f = @(x) cos(x)+2*sin(x);  % signal
a=0; b=10;         % Interval of interest is [a,b]
h = 0.01;          % Fine spacing for plots and power computations
t=a:h:b;           % Equispaced evaluation points in [a,b] spaced h apart
x=a:b;             % Interpolation abscissas, every integer in [a,b]
y=f(x);            % Interpolation ordinates for this signal
fsig = f(t);                    % Signal, at many points in [a,b]
fpwc = interp1(x,y,t,"nearest"); % Sampled pw constant on [k-1/2,k+1/2]
fpwl = interp1(x,y,t,"linear");  % Sampled pw linear on [k,k+1]
fcsp = interp1(x,y,t,"spline");  % Cubic spline interpolation, nodes k

% Signal to noise ratio, approximated using evaluation points in t
sigpow = norm(fsig);    % square root of power
snrpwc = 20*log10(sigpow/norm(fsig-fpwc)) % (a) pw constant:  10.822
snrpwl = 20*log10(sigpow/norm(fsig-fpwl))  % (b) pw linear:  20.989
snrcsp = 20*log10(sigpow/norm(fsig-fcsp))  % (c) cubic spline:  40.461
```

□

6. [10 points] Let $f(x) = \cos(x) + 2\sin(x)$ for $0 \le x \le 10$. Note that $f \in L^2([0, 10])$.

   Let s be the band-limited approximation to f with bandwidth 1, namely

$$s(x) = \sum_{n=0}^{10} f(n)\mathrm{sinc}\,(x - n).$$

   Estimate the signal-to-noise ratio in decibels for this approximation as in the previous problem,

   Hint: Octave has a built-in `sinc()`. Use it in your own function for s, then compute $\|f\|^2$, $\|s\|^2$, and $\|f - s\|^2$ as sums of squares at the evaluation points $\{0, 0.01, 0.02, \ldots, 10\}$.

Solution: Use the Octave commands from the previous exercise and define the sampled bandlimited approximation as follows:

```
fbl1 = 0*t;  % initialize a vector of values for the bandlimited approximation
for (k=a:b) fbl1 = fbl1 + f(k)*sinc(t-k);  end
snrbl1 = 20*log10(sigpow/norm(fsig-fbl1)) % bandlimited approx: 24.166
```

□

7. Let $f = f(x, y)$ be the joint probability density supported on the region $R = \{(x, y) : 0 \le x \le 1, x-1 \le y \le x+1\}$ and defined by the formula $f(x, y) = 1 - |y - x|$ for $(x, y) \in R$, with $f(x, y) = 0$ elsewhere.
   (a) [5 points] Show that $\iint_R f(x, y)\, dxdy = 1$.
   (b) [5 points] Compute the normalizing constant c_x and determine $f(y \,|\, x)$.
   (c) [5 points] Compute the expectation $E(y \,|\, x)$. Is $d(x) = x$ an unbiased estimator?
   (d) [5 points] Compute the risk $R(d, y)$ for the decision function $d(x) = x$. Does it depend on y?

Solution: (a) The integral is the volume of the skewed prism under the graph of f, which has a constant cross-sectional area 1 at each x, consisting of a triangle of height 1 and base 2. The volume is that cross-sectional area times the length of the x-interval, which is also 1. (This proof uses a version of the Theorem of Pappus.)

(b) Using Equation 4.38, compute

$$c_x = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{x-1}^{x+1} (1 - |y - x|)\, dy = 1, \quad 0 \le x \le 1,$$

which is independent of x since for every x the integrand is a hat function whose graph bounds an isosceles triangle of base 2 and height 1. Thus

$$f(y \,|\, x) = \frac{1}{c_x} f(x, y) = \begin{cases} 1 - |y - x|, & \text{if } 0 \le x \le 1 \text{ and } x - 1 \le y \le x + 1; \\ 0, & \text{otherwise.} \end{cases}$$

(c) Using the result from part b, compute the expectation

$$\begin{aligned} E(y \,|\, x) &= \int_{-\infty}^{\infty} y f(y \,|\, x)\, dy = \int_{x-1}^{x+1} y(1 - |y - x|)\, dy = \int_{-1}^{+1} [x + u](1 - |u|)\, du \\ &= x \int_{-1}^{+1} (1 - |u|)\, du + \int_{-1}^{+1} u(1 - |u|)\, du = x + 0 = x. \end{aligned}$$

The third step follows from the substitution $u = y - x$, so $du = dy$ and $y = x + u$. At the fourth step, the integral of $u(1 - |u|)$ on $[-1, 1]$ vanishes by antisymmetry, and the integral of the hat function $1 - |x|$ on $[-1, 1]$ is 1 by the reasoning in part a. Thus $E(y \,|\, x) = x$, so the decision function $d(x) = x$ gives an unbiased estimator.

(d) Using Equation 4.43, compute

$$\begin{aligned} R^2(d, y) &= \int_0^1 |d(x) - y|^2 f(y \,|\, x)\, dy = \int_{x-1}^{x+1} |x - y|^2 (1 - |y - x|)\, dy \\ &= \int_{-1}^{+1} |u|^2 (1 - |u|)\, du = 2 \int_0^1 (u^2 - u^3)\, du = \frac{1}{6}. \end{aligned}$$

The third step follows from the substitution $u = x + y$. The risk function $R(d, y) = 1/\sqrt{6}$ is evidently independent of the ideal value y. □