Ma 450: Mathematics for Multimedia

Solution: to Homework Assignment 5

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Due Sunday, April 16th, 2023

All solutions are worth 10 points.

1. Draw the graphs of $w(t)$, $w(t/2)$, and $w(3t)$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

   Solution: The graphs are shown in Figure 1. □

2. Draw the graphs of $w(\frac{t}{3}-4)$ and $w(\frac{t-4}{3})$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.

   Solution: The graphs are shown in Figure 2. □

3. Let $f = f(\mathbf{a}) = f(a,b)$ be the function on **Aff** defined by $f(\mathbf{a}) = \mathbf{1}_D(\mathbf{a})$, where $\mathbf{1}_D$ is the indicator function of the region $D = \{\mathbf{a} = (a,b) : A < a < A', B < b < B'\} \subset \mathbf{Aff}$ for $0 < A < A'$ and $-\infty < B < B' < \infty$. Evaluate $\int_{\mathbf{Aff}} f(\mathbf{a})\, d\mathbf{a}$ using the normalized left-invariant integral on **Aff**.

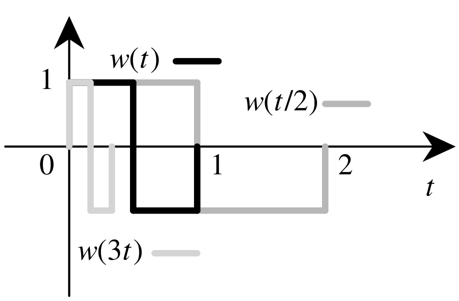

Figure 1: Graphs of $w(t)$, $w(t/2)$, and $w(3t)$ for the Haar function w.

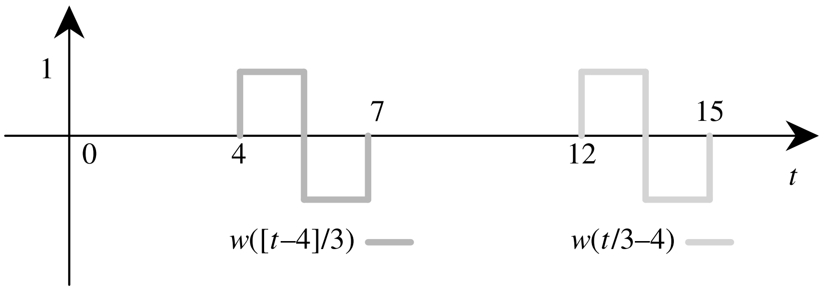

Figure 2: Graphs of $w(\frac{t}{3}-4)$ and $w(\frac{t-4}{3})$ for the Haar function w.

Solution: Use the integral defined in Equation 5.19:

$$\begin{aligned}\int_{\mathbf{Aff}} f(\mathbf{a})\,d\mathbf{a} &\stackrel{\text{def}}{=} \int_{b=-\infty}^{\infty}\int_{a=0}^{\infty} f(a,b)\,\frac{dadb}{a^2} = \int_{b=B}^{B'}\int_{a=A}^{A'} \frac{1}{a^2}\,dadb \\ &= (B'-B)\int_{a=A}^{A'}\frac{1}{a^2}\,da = (B'-B)\left(\frac{1}{A}-\frac{1}{A'}\right).\end{aligned}$$

□

4. Let $w = w(t)$ be the Haar mother function and define

$$\phi^J_{M,K}(t) \stackrel{\text{def}}{=} \sum_{j=M+1}^{M+J} \frac{1}{2^j} w\left(\frac{t-K}{2^j}\right)$$

for arbitrary fixed $K \in \mathbf{R}$ and $M, J \in \mathbf{Z}$ with $J > 0$.

a. Show that

$$\lim_{J\to\infty} \phi^J_{M,K}(t) = 2^{-M}\mathbf{1}_{[K,K+2^M)}(t) \stackrel{\text{def}}{=} \phi_{M,K}(t),$$

b. Show that $\left\langle \phi^J_{M,K}, u\right\rangle \to \langle \phi_{M,K}, u\rangle$ as $J \to \infty$ for any function $u \in L^2(\mathbf{R})$.

(Hint: use Equation 5.4 and Lemma 5.1.)

Solution:

a. Using ϕ^J as defined in Equation 5.4, evaluate

$$\begin{aligned}\phi^J_{M,K}(t) &= \sum_{j=M+1}^{M+J} \frac{1}{2^j} w\left(\frac{t-K}{2^j}\right) = \sum_{j=1}^{J} \frac{1}{2^{j+M}} w\left(\frac{t-K}{2^{j+M}}\right) \\ &= \frac{1}{2^M}\sum_{j=1}^{J}\frac{1}{2^j} w\left(\frac{1}{2^j}\frac{t-K}{2^M}\right) = \frac{1}{2^M}\phi^J\left(\frac{t-K}{2^M}\right).\end{aligned}$$

Thus for all $t \in \mathbf{R}$,

$$\lim_{J\to\infty} \phi^J_{M,K}(t) = \frac{1}{2^M} \lim_{J\to\infty} \phi^J(\frac{t-K}{2^M}) = \frac{1}{2^M}\mathbf{1}(\frac{t-K}{2^M}) = 2^{-M}\mathbf{1}_{[K,K+2^M)}(t).$$

b. Lemma 5.1 then gives

$$\lim_{J\to\infty} \left\langle \phi^J_{M,K}, u \right\rangle = \frac{1}{2^M}\int_{-\infty}^{\infty} \mathbf{1}(\frac{t-K}{2^M})u(t)\,dt = \frac{1}{2^M}\int_K^{K+2^M} u(t)\,dt = \langle \phi_{M,K}, u\rangle\,,$$

as claimed. □

5. Compute $\|w\|$, where

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log\xi)^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \le 0. \end{cases}$$

(Hint: use Plancherel's theorem and Equation B.6 in Appendix B.)

Solution: By Plancherel's theorem, $\|w\| = \|\mathcal{F}w\|$. Substitute $\xi \leftarrow e^{\eta+\frac{1}{4}}$ to compute

$$\|\mathcal{F}w\|^2 = \int_{\xi=0}^{\infty} e^{-2(\log\xi)^2}\,d\xi = \int_{\eta=-\infty}^{\infty} e^{-2(\eta+\frac{1}{4})^2} e^{\eta+\frac{1}{4}}\,d\eta = e^{1/8}\int_{\eta=-\infty}^{\infty} e^{-2\eta^2}\,d\eta.$$

Finally, substitute $x \leftarrow \eta\sqrt{\frac{2}{\pi}}$ into Equation B.6, $\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$, to get $\int_{-\infty}^{\infty} e^{-2\eta^2}\,d\eta = \sqrt{\frac{\pi}{2}}$. Thus $\|w\| = \sqrt{\|\mathcal{F}w\|^2} = e^{1/16}\left(\frac{\pi}{2}\right)^{1/4} \approx 1.1917$.
NOTE: The norm may also be computed with Macsyma as follows:

```
sqrt(integrate(exp(-log(x)**2)**2,x,0,inf));
```

A numerical approximation may then be found using the `float()` command. □

6. Let w be the function defined by

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log|\xi|)^2}, & \text{if } \xi \ne 0; \\ 0, & \text{if } \xi = 0. \end{cases}$$

Show that w is admissible and compute its normalization constant c_w.

Solution: First note that $\mathcal{F}w(-\xi) = \mathcal{F}w(\xi)$, so $|\mathcal{F}w(-\xi)|^2 = |\mathcal{F}w(\xi)|^2$. Thus if the $+\xi$ admissibility integral exists, then the $-\xi$ integral exists as well and has the same value.

Next, compute the $+\xi$ admissibility integral:

$$c_w = \int_0^\infty \frac{|\mathcal{F}w(\xi)|^2}{\xi}\,d\xi = \int_{\xi=0}^\infty \frac{e^{-2(\log\xi)^2}}{\xi}\,d\xi = \int_{\eta=-\infty}^\infty \frac{e^{-2\eta^2}}{e^\eta}e^\eta\,d\eta = \int_{\eta=-\infty}^\infty e^{-2\eta^2}\,d\eta.$$

This follows from the substitution $\xi \leftarrow e^\eta$. Finally, substitute $x \leftarrow \eta\sqrt{\frac{2}{\pi}}$ into Equation B.6, $\int_{-\infty}^\infty e^{-\pi x^2}\,dx = 1$, to get $c_w = \int_{-\infty}^\infty e^{-2\eta^2}\,d\eta = \sqrt{\frac{\pi}{2}} \approx 1.2533$. Thus w is admissible.

NOTE: This integral may also be computed with Macsyma:

```
integrate(exp(-log(x)**2)**2/x,x,0,inf);
```

A numerical approximation may then be found using the `float()` command. □

7. Fix $A < 0$, $B > 0$, and $R > 1$ and suppose that $w = w(x)$ is a function satisfying $\mathcal{F}w(\xi) = 1$ if $RA < \xi < A$ or $B < \xi < RB$, with $\mathcal{F}w(\xi) = 0$, otherwise.

   a. Show that w satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant c_w.

   b. Give a formula for w.

Solution:

a. Plancherel's theorem guarantees that w belongs to $L^2(\mathbf{R})$, since $\|w\| = \|\mathcal{F}w\| = \sqrt{(B-A)(R-1)} < \infty$.

Compute the two admissibility integrals:

$$\begin{aligned}
\int_0^\infty \frac{|\mathcal{F}w(-\xi)|^2}{\xi}\,d\xi &= \int_{-A}^{-RA} \frac{1}{\xi}\,d\xi = \log(-RA) - \log(-A) = \log R;\\
\int_0^\infty \frac{|\mathcal{F}w(\xi)|^2}{\xi}\,d\xi &= \int_{B}^{RB} \frac{1}{\xi}\,d\xi = \log(RB) - \log(B) = \log R.
\end{aligned}$$

These are finite and equal for $R > 1$, so w is admissible with normalization constant $c_w = \log R$.

b. The inverse Fourier integral transform of $\mathcal{F}w$ gives the formula for w:

$$w(x) = \int_{-\infty}^\infty e^{2\pi i x\xi}\mathcal{F}w(\xi)\,d\xi = \int_{RA}^{A} + \int_B^{RB} e^{2\pi i x\xi}\,d\xi = \frac{e^{2\pi i xA} - e^{2\pi i xRA} + e^{2\pi i xRB} - e^{2\pi i xB}}{2\pi i x}.$$

Note that $w(0) \stackrel{\text{def}}{=} (B-A)(R-1)$ makes the function continuous, in fact smooth, by L'Hôpital's rule. Further simplification is not necessary. □

8. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the antisymmetry condition $h(0) = -h(3)$ and $h(1) = -h(2)$, or prove that none exist.

   Solution: None exist. Antisymmetry would violate the normalization condition $h(0) + h(2) = \frac{1}{\sqrt{2}} = h(1) + h(3)$. Thus no antisymmetric real-valued orthogonal low-pass CQFs of length 4 exist. □

9. Find a real-valued orthogonal low-pass CQF of length 4 satisfying the symmetry condition $h(0) = h(3)$ and $h(1) = h(2)$, or prove that none exist.

   Solution: Only one exists and it is degenerate in that it only has 2 nonzero coefficients. Let h be an orthogonal CQF with nonzero real coefficients $h(0)$, $h(1)$, $h(2)$, and $h(3)$. Then h must be of the form

$$h(0) = \frac{1-c}{\sqrt{2}(1+c^2)}; \quad h(1) = \frac{1+c}{\sqrt{2}(1+c^2)}; \quad h(2) = \frac{c(c+1)}{\sqrt{2}(1+c^2)}; \quad h(3) = \frac{c(c-1)}{\sqrt{2}(1+c^2)},$$

   where c is some real number different from 0 and ± 1. The symmetry conditions imply $1 - c = c(c-1)$ and $1 + c = c(c+1)$, which implies $c^2 = 1$ and thus $c = \pm 1$. But $c = 1$ leads to a filter of length 2, the Haar filter. The other choice $c = -1$ leads to the unique symmetric real-valued orthogonal CQF of length 4: $\{1/sqrt2, 0, 0, 1/\sqrt{2}\}$. □

10. Suppose that an orthogonal MRA has a scaling function ϕ satisfying $\phi(t) = 0$ for $t \notin [a, b]$. Prove that the low-pass filter h for this MRA must satisfy $h(n) = 0$ for all $n \notin [2a - b, 2b - a]$. (This makes explicit the finite support of h in Equation 5.36.)

    Solution: If $\phi(t) = 0$ for $t \notin [a, b]$, then $\phi(2t - n) = 0$ for $t \notin \left[\frac{a+n}{2}, \frac{b+n}{2}\right]$. Use the orthonormality of $\{\sqrt{2}\phi(2t - k) : k \in \mathbf{Z}\}$ in V_{-1} to compute

$$\frac{1}{\sqrt{2}} h(n) = \left\langle \phi(2t - n),\ \sum_k h(k)\sqrt{2}\,\phi(2t - k) \right\rangle = \langle \phi(2t - n), \phi(t) \rangle .$$

    But the support intervals $\left[\frac{a+n}{2}, \frac{b+n}{2}\right]$ and $[a, b]$ of the two factors in the inner product will not overlap if $(a + n)/2 > b \iff n > 2b - a$ or if $(b + n)/2 < a \iff n < 2a - b$. Thus $h(n) = 0$ if $n \notin [2a - b, 2b - a]$. □

11. Suppose that $h = \{h(k) : k \in \mathbf{Z}\}$ and $g = \{g(k) : k \in \mathbf{Z}\}$ satisfy the orthogonal CQF conditions. Show that the 2-periodizations h_2, g_2 of h and g are the Haar filters. Namely, show that $h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2}$.

Solution: Use the normalization conditions for h and g to evaluate the 2-periodization formula:

$$h_2(0) = \sum_k h(0+2k) = 1/\sqrt{2}; \qquad h_2(1) = \sum_k h(1+2k) = 1/\sqrt{2};$$
$$g_2(0) = \sum_k g(0+2k) = 1/\sqrt{2}; \qquad g_2(1) = \sum_k g(1+2k) = -1/\sqrt{2}.$$

Since h_2 and g_2 are 2-periodic, this determines all their values. □

12. Let ϕ be the scaling function of an orthogonal MRA, and let ψ be the associated mother function. For $(x,y) \in \mathbf{R}^2$, define

$$e_0(x,y) = \phi(x)\phi(y), \qquad e_1(x,y) = \phi(x)\psi(y)$$
$$e_2(x,y) = \psi(x)\phi(y), \qquad e_3(x,y) = \psi(x)\psi(y).$$

Prove that the functions $\{e_n : n = 0,1,2,3\}$ are orthonormal in $L^2(\mathbf{R}^2)$, the inner product space of square-integrable functions on $\mathbf{R}^2$.

Solution: First note that ϕ and ψ are compactly supported, so e_0, e_1, e_2, e_3 vanish outside some closed and bounded rectangles in $\mathbf{R}^2$. Also, both ϕ and ψ are integrable and square-integrable as functions of one variable, so e_0, e_1, e_2, e_3 are integrable by iterating one-dimensional integrals. Write $e_i^1(x)$ for the x-dependent factor of $e_i(x,y)$ and $e_i^2(y)$ for the y-dependent factor of $e_i(x,y)$; then $e_0^1 = e_1^1 = e_0^2 = e_2^2 = \phi$, while $e_2^1 = e_3^1 = e_1^2 = e_3^2 = \psi$. Thus the inner products in $L^2(\mathbf{R}^2)$ are computable as follows:

$$\begin{aligned} \langle e_i, e_j \rangle &= \int_{\mathbf{R}^2} e_i(x,y) e_j(x,y)\, dxdy \\ &= \left(\int_{\mathbf{R}} e_i^1(x) e_j^1(x)\, dx \right) \left(\int_{\mathbf{R}} e_i^2(y) e_j^2(y)\, dy \right) \\ &= \left\langle e_i^1, e_j^1 \right\rangle \left\langle e_i^2, e_j^2 \right\rangle, \qquad i,j \in \{0,1,2,3\}. \end{aligned}$$

But if $i \neq j$, then at least one of these inner products is $\langle \phi, \psi \rangle$ or $\langle \psi, \phi \rangle$, which are both zero since $\phi \perp \psi$. Hence $\{e_0, e_1, e_2, e_3\}$ is an orthogonal set in $L^2(\mathbf{R}^2)$.

On the other hand, if $i = j$, then the two inner products are both 1 since $\|\phi\| = \|\psi\| = 1$. Hence $\{e_0, e_1, e_2, e_3\}$ is an orthonormal set in $L^2(\mathbf{R}^2)$. □