If $x = (x_1, \ldots, x_n)$, and $F = F(x)$ is a differentiable function of several variables with gradient $\nabla F(x) = (\frac{\partial F}{\partial x_1}(x), \ldots \frac{\partial F}{\partial x_n}(x))$, then by the Mean Value Theorem we have

$$F(x) - F(v(x)) = \nabla F(z) \cdot \Delta x, \tag{1.10}$$

for some z on the line segment connecting x and $v(x) \stackrel{\text{def}}{=} (v(x_1), \ldots, v(x_n))$. Here Δx has components $\Delta x_k = x_k - v(x_k)$ satisfying $|\Delta x_k| \leq |x_k|\epsilon_f$. Writing M_k for the maximum value of $|\frac{\partial F}{\partial x_k}(z)|$ over all z between x and $v(x)$, we have

$$|F(x) - F(v(x))| \leq \epsilon_f \sum_{k=1}^{n} M_k |x_k|,$$

by the triangle inequality. The rest of the argument is similar to the one-variable case and gives the inequality

$$\left| \frac{F(x) - v(F(v(x)))}{F(x)} \right| < \left(1 + \frac{\sum_k M_k |x_k|}{|F(x)|} \right) \epsilon_f. \tag{1.11}$$

This inequality applies to products and quotients. For example, let $n = 2$ and put $F(x) = x_1 x_2$. Then $\nabla F(x) = (x_2, x_1)$, so $M_1 \approx |x_2|$ and $M_2 \approx |x_1|$ for z within $\epsilon_f \ll 1$ of x. The right-hand side of Inequality 1.11 simplifies to

$$\left(1 + \frac{|x_2 x_1| + |x_1 x_2|}{|x_2 x_1|} \right) \epsilon_f = 3\epsilon_f,$$

as in Inequality 1.6. The case of quotients is left as an exercise.

1.3 Exercises

1. Suppose a divides b and b divides a. Must $a = b$?

2. Write a computer program that finds the greatest common divisor of two integers a and b, assuming $b > a > 0$.

3. Prove that distinct primes are relatively prime.

4. Find the greatest common divisor of the three numbers 299 792 458, 6 447 287, and 256 964 964.

5. Find the quasi-inverse of 2301 modulo 19 687. (Hint: implement the extended Euclid algorithm first.)

6. Prove that integer overflow or underflow occurs in w-bit twos complement integer arithmetic if and only if the carry into the sign bit is different from the carry out of the sign bit.

7. Express the integer 14 600 926 (base 10) in hexadecimal.

8. Prove that if $p \in \mathbf{Z}$ is a prime number, then $\sqrt{p}$ is not a rational number.

9. Write a computer program to read an integer in decimal notation and then print its binary digits and its hexadecimal digits. (Hint: most computers expect decimal number inputs and thus have built-in functions to read them.)

10. Convert the approximation $\pi \approx 3.1415926535897932$ (base 10) into the nearest 8-digit hexadecimal fraction.

11. Using 52 bits to represent the mantissa in IEEE binary floating-point format, how many decimal digits of accuracy are obtained?

12. What will $\sum_{k=1}^{10^8} 1.0$ equal on the example computer on p. 15, which uses IEEE 32-bit floating-point arithmetic?

13. Write a program to read 32-bit IEEE binary floating-point format and print the number in scientific notation. Have it treat NaN, $\pm\infty$, and ± 0 properly and have it signal when the number is subnormal.

14. Derive Inequality 1.7 from Inequality 1.11.

15. Determine and prove whether the following computations are well-conditioned or ill-conditioned:
    a. $(x, y) \mapsto \sqrt{x^2 + y^2}$, for $x \neq 0$ and $y \neq 0$
    b. $x \mapsto x \log x$, for $x > 0$
    c. $x \mapsto \lfloor x \rfloor$

1.4 Further Reading

- ANSI/IEEE. *Standard for Binary Floating-Point Arithmetic.* Document 754-1985, catalog number SH 10116-NYF. ISBN 1-55937-653-8.

- Donald Knuth. *Fundamental Algorithms*, volume 1. Addison-Wesley, Reading, Massachusetts, second edition, 1973. ISBN 0-201-03809-9.

- Behrooz Parhami. *Computer Arithmetic: Algorithms and Hardware Designs.* Oxford University Press, New York, 2000. ISBN 0-19-512583-5.

- Herbert Schildt. *The Annotated ANSI C Standard: ANSI/ISO 9899-1990.* Osborne Mcgraw Hill, Berkeley, California, 1993. ISBN 0-07-881952-0.

- Douglas R. Stinson. *Cryptography: Theory and Practice.* CRC Press, Boca Raton, Florida, 1995. ISBN 0-8493-8521-0.