

Independence: $GH^* = HG^* = 0 \in \mathbf{Mat}(M \times M)$. Thus the column space of G^* is orthogonal to the column space of H^* . As a consequence, for all $\mathbf{x} \in \mathbf{E}^{2M}$, we have $G^*G\mathbf{x} \perp H^*H\mathbf{x}$.

Completeness: $2H^*H + \frac{1}{2}G^*G = Id \in \mathbf{Mat}(2M \times 2M)$. Thus every $\mathbf{x} \in \mathbf{E}^{2M}$ may be written as $\mathbf{x} = \mathbf{s} + \mathbf{d}$, for some \mathbf{s} in the M -dimensional column space of H^* and some \mathbf{d} in the M -dimensional column space of G^* .

Running Averages. Fix $K > 1$ and for $0 \leq m \leq N - K$ and $0 \leq n < N$ let

$$A(m, n) = \begin{cases} \frac{1}{K}, & \text{if } m \leq n < m + K; \\ 0, & \text{otherwise.} \end{cases} \quad (2.65)$$

Then $A\mathbf{x}(m)$ is the average of the K components of \mathbf{x} starting with $x(m)$. The associated matrix is

$$\begin{pmatrix} \frac{1}{K} & \frac{1}{K} & \cdots & \frac{1}{K} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{K} & \cdots & \frac{1}{K} & \frac{1}{K} & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{K} & \frac{1}{K} & \cdots & \frac{1}{K} \end{pmatrix} = \frac{1}{K} \sum_{m \in M} \sum_{n=m}^{m+K-1} \mathbf{e}_{m,n}. \quad (2.66)$$

As usual, out-of-range coefficients are set to zero.

2.3 Exercises

- How many vertices and edges are there in the 5-cube, the unit cube in Euclidean 5-space? Find a formula in terms of N for the number of vertices and edges of an N -cube.
- Find an example subspace of \mathbf{R}^N of dimension k for each $k = 0, 1, \dots, N$.
- Show that the system of inequalities 2.14, 2.15, and 2.16 is sharp for every N by finding example vectors in \mathbf{C}^N that give equality.
- Prove that the following are equivalent:
 - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ and $\|a\mathbf{x}\| = |a| \|\mathbf{x}\|$, for all vectors \mathbf{x}, \mathbf{y} and scalars a ;
 - $\|a\mathbf{x} + b\mathbf{y}\| \leq |a| \|\mathbf{x}\| + |b| \|\mathbf{y}\|$, for all vectors \mathbf{x}, \mathbf{y} and scalars a, b .
- Show that for any subset \mathbf{Y} of an inner product space \mathbf{X} , we have $\mathbf{Y} \cap \mathbf{Y}^\perp = \{\mathbf{0}\}$, and $\mathbf{Y} \subset (\mathbf{Y}^\perp)^\perp$. (This is Lemma 2.5.)
- Suppose that $\mathbf{Y} = \text{span}\{\mathbf{y}_n : n = 1, \dots, N\}$. Show that if $\langle \mathbf{x}, \mathbf{y}_n \rangle = 0$ for all n , then $\mathbf{x} \in \mathbf{Y}^\perp$. (This is Lemma 2.6.)