Finally, observe that

$$r\left(\frac{t-\alpha}{\epsilon}\right) = \begin{cases} 1, & \text{if } t \ge \alpha+\epsilon, \\ 0, & \text{if } t \le \alpha-\epsilon, \end{cases} \qquad r\left(\frac{\alpha-t}{\epsilon}\right) = \begin{cases} 0, & \text{if } t \ge \alpha+\epsilon, \\ 1, & \text{if } t \le \alpha-\epsilon. \end{cases}$$

These valuations result in the formula in Equation 3.20.

d. Performing the previous calculation, but starting from Equation 3.15, we get

$$\begin{aligned} S(r,\alpha,\epsilon)u(t) &= \tau_\alpha \delta_\epsilon \, S \, \delta_\epsilon^{-1} \tau_\alpha^{-1} u(t) \\ &= \begin{cases} \bar{r}(\frac{t-\alpha}{\epsilon})u(t) - r(\frac{\alpha-t}{\epsilon})u(2\alpha - t), & \text{if } \alpha < t \le \alpha+\epsilon, \\ r(\frac{\alpha-t}{\epsilon})u(t) + \bar{r}(\frac{t-\alpha}{\epsilon})u(2\alpha-t), & \text{if } \alpha-\epsilon \le t < \alpha, \\ u(t), & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, since the inverse of $\tau_\alpha\delta_\epsilon$ is $\delta_\epsilon^{-1}\tau_\alpha^{-1} = \delta_{1/\epsilon}\tau_{-\alpha}$, we can simplify

$$\begin{aligned} S(r,\alpha,\epsilon)F(r,\alpha,\epsilon) &= \left(\tau_\alpha\delta_\epsilon \, S \, \delta_\epsilon^{-1}\tau_\alpha^{-1}\right)\left(\tau_\alpha\delta_\epsilon \, F \, \delta_\epsilon^{-1}\tau_\alpha^{-1}\right) \\ &= \tau_\alpha\delta_\epsilon \, SF \, \delta_\epsilon^{-1}\tau_\alpha^{-1} = \tau_\alpha\delta_\epsilon\delta_\epsilon^{-1}\tau_\alpha^{-1} = Id, \end{aligned}$$

since $SF = Id$. The other order, $F(r,\alpha,\epsilon)S(r,\alpha,\epsilon) = Id$, likewise follows from $FS = Id$. □

3. **Solution:** Suppose that r is a rising cut-off function and $\epsilon < T/2$. Write $F_0 = F(r,0,\epsilon)$ and $F_T = F(r,T,\epsilon)$ for the fraying operators at 0 and T, respectively. Since $u(t+T) = u(t)$ for all t, the function $v \stackrel{\text{def}}{=} F_0F_Tu$ satisfies $v(t+T) = v(t)$ for all $-\frac{T}{2} \le t \le \frac{T}{2}$. In particular, that means $v_I(t) = v(t)$ for all $-\epsilon \le t \le T+\epsilon$. Also, $v(t) = u(t)$ if t lies between ϵ and $T-\epsilon$, outside the reach intervals of F_0 and F_T.

   Now put $I = [0,T]$. The second formula for the loop operator, Equation 3.23, simplifies to

   $$L_Iv(t) = \begin{cases} S(r,0,\epsilon)v(t), & \text{if } 0 < t \le \epsilon; \\ S(r,T,\epsilon)v(t), & \text{if } T-\epsilon \le t < T; \\ v(t), & \text{otherwise.} \end{cases}$$

   Writing $S_0 = S(r,0,\epsilon)$ and $S_T = S(r,T,\epsilon)$ for the splicing operators at 0 and T, one calculates

   $$L_Iv(t) = \begin{cases} S_0F_0u(t) = u(t), & \text{if } 0 < t \le \epsilon, \\ S_TF_Tu(t) = u(t), & \text{if } T-\epsilon \le t < T, \\ v(t) = u(t), & \text{if } \epsilon \le t \le T-\epsilon. \end{cases}$$

   Thus $L_IF_0F_Tu = u$ on I, so $(L_IF_0F_Tu)_I = u_I = u$ on $\mathbf{R}$. □

4. **Solution:** (i) $t > 1$ implies $\tilde{F}u(t) = \bar{1}u(t) = u(t)$ and $\tilde{S}u(t) = 1u(t) = u(t)$. Likewise, $t < -1$ implies $\tilde{F}u(t) = 1u(t) = u(t)$ and $\tilde{S}u(t) = \bar{1}u(t) = u(t)$.