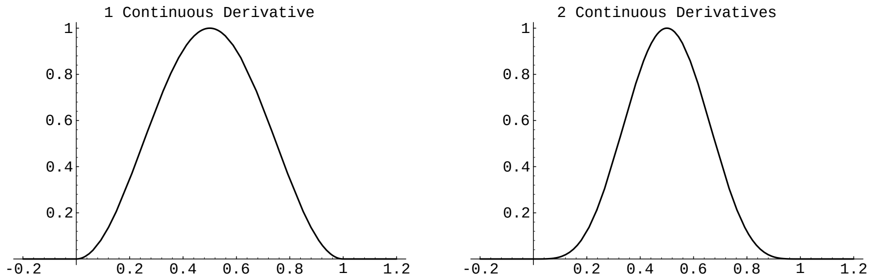

Figure 3.1: Cosine-based window functions with 1 and 2 continuous derivatives

In other words, localization to I multiplies f by $\mathbf{1}_I$, the indicator function of the interval. The definition may also be applied to other kinds of intervals, like $I = (a, b)$ or $I = [a, b)$.

In general, a function $f = f(t)$ is said to have *compact support* if there is some closed and bounded interval $I = [a, b]$ such that $f(t) = 0$ for all $t \notin I$. Such an f is said to be *supported on* I. A simple example is the zero function, which has compact support using any I. Less special is $\mathbf{1}$, which has compact support and is supported on $[0, 1]$. The function $g(t) = e^{-t^2}$, which is never zero and therefore not compactly supported, may be described as supported on all of $\mathbf{R}$. But the restriction of any function f to a bounded interval I will have compact support, and $\mathbf{1}_I f$ will be supported in I.

Even if f is continuous, its localization need not be: there may be jump discontinuities at the endpoints a and b of I. But if $f(a) = f(a+) = 0$ and $f(b) = f(b-) = 0$ and f is continuous at each $t \in (a, b)$, then $\mathbf{1}_I f$ is continuous. If in addition f satisfies a Lipschitz condition on I, then the localization $\mathbf{1}_I f$ will satisfy a Lipschitz condition on all of $\mathbf{R}$.

A more sophisticated solution is to smooth the cut by pinching the ends down to zero. Let $u = u(t)$ be a "window function" such as the one defined below, which is called a *Hanning window*:

$$u(t) = \begin{cases} 0, & \text{if } t < 0 \text{ or } t > 1; \\ \frac{1}{2} - \frac{1}{2}\cos 2\pi t, & \text{if } 0 \le t \le 1. \end{cases} \tag{3.5}$$

its graph is plotted in the left half of Figure 3.1. This function u has one continuous derivative: $u'(0+) = 0 + \pi \sin(2\pi 0+) = 0 = u'(0-)$ and $u'(1-) = 0 + \pi \sin(2\pi-) = 0 = u'(1+)$, so the derivatives of the pieces match up at $t = 0$ and $t = 1$.

The Hanning window is one member of a family of functions of the form

$$u(t) = \begin{cases} A - B\cos 2\pi t + C\cos 4\pi t, & \text{if } t \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Such a u can be made smoother by choosing $C \neq 0$ as follows: note that for all windows in the family, $u(0+) = A - B + C = u(1-)$, $u'(0+) = u'(1-) = 0$, and

$$u''(0+) = 4\pi^2 B - 16\pi^2 C = u''(1-).$$