

$\xi$ . Thus, it is legal to differentiate once or twice with respect to  $a$  or  $b$  under the integral sign:

$$\begin{aligned}\frac{\partial}{\partial a} W u(a, b) &= \int_{-\infty}^{\infty} \xi e^{-2\pi i b \xi} \phi'(a\xi) \mathcal{F}u(\xi) d\xi; \\ \frac{\partial}{\partial b} W u(a, b) &= -2\pi i \int_{-\infty}^{\infty} \xi e^{-2\pi i b \xi} \phi(a\xi) \mathcal{F}u(\xi) d\xi.\end{aligned}$$

It remains to show that these derivatives are continuous functions of  $a, b$  away from the line  $a = 0$ . But in both cases, this follows from the observation that the integrands are continuous functions of  $a, b$ .  $\square$

8. **Solution:** Since  $w = \mathcal{F}\mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}$ , use Plancherel's theorem to compute  $\|w\| = \|\mathcal{F}\mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}\| = \|\mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]\| = 1$ .  $\square$

9. **Solution:** By the previous solution and by combining integrals, calculate that  $\mathcal{F}w = \mathbf{1}_{[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]}$ . Thus,

$$c_w = \int_0^{\infty} \frac{|\mathbf{1}_{[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]}(\xi)|^2}{\xi} d\xi = \int_{\frac{1}{2}}^1 \frac{d\xi}{\xi} = \log 2 \approx 0.69315 < \infty.$$

But  $\mathcal{F}w(-\xi) = \mathcal{F}w(\xi)$ , so the  $-\xi$  integral is the same, so  $w$  is admissible.  $\square$

10. **Solution:** The Fourier integral transform of  $w$  is

$$\int_{-\infty}^{\infty} e^{-2\pi i x \xi} w(x) dx.$$

Since  $w(x) = 1$  if  $0 < x < \frac{1}{2}$  and  $w(x) = -1$  if  $\frac{1}{2} < x < 1$ , that simplifies to

$$\int_0^{\frac{1}{2}} e^{-2\pi i x \xi} - \int_{\frac{1}{2}}^1 e^{-2\pi i x \xi} = \frac{(e^{-\pi i \xi} - 1)^2}{2\pi i \xi}.$$

$\square$

11. **Solution:** It is necessary to show that  $\langle \phi_j, \phi_k \rangle = \delta(j - k)$ . But Plancherel's theorem allows writing

$$\langle \phi_j, \phi_k \rangle = \langle \mathcal{F}\phi_j, \mathcal{F}\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i(k-j)\xi} d\xi = \delta(j - k),$$

since  $\mathcal{F}\phi_k(\xi) = e^{2\pi i k \xi} \mathcal{F}\text{sinc}(\xi) = e^{2\pi i k \xi} \mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}(\xi)$ .  $\square$

12. **Solution:** Show that  $\sum_k g(2k) = -\sum_k g(2k + 1) = \frac{1}{\sqrt{2}}$ :

$$\begin{aligned}\sum_k g(2k) &= \sum_k (-1)^{2k} \overline{h(2M - 1 - 2k)} = \sum_k \overline{h(2M - 1 - 2k)} \\ &= \sum_k \overline{h(2(M - k) - 1)} = \sum_l \overline{h(2l + 1)} = \frac{1}{\sqrt{2}},\end{aligned}$$