11. Prove that the set of functions $\{\phi_k : k \in \mathbf{Z}\}$ defined by $\phi_k(t) = \operatorname{sinc}(t-k)$ is orthonormal. (Hint: use Plancherel's theorem and the fact that $\mathcal{F}\mathrm{sinc} = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$.)

12. Show that if $h = \{h(k) : k \in \mathbf{Z}\}$ is the low-pass filter from an MRA, and M is any fixed integer, then defining
$$g(k) = (-1)^k \overline{h(2M-1-k)}, \qquad \text{for all } k \in \mathbf{Z}, \tag{5.76}$$
gives another high-pass filter for that MRA.

13. Show that the Daubechies 4 low-pass filter of Equation 5.49 is derived from the general form in Equation 5.48 using the parameter value $c = 2 - \sqrt{3}$.

14. Suppose that h, g is a CQF pair derived from an orthogonal MRA with scaling function ϕ and mother function ψ. Suppose that $u \subset L^2(\mathbf{R})$ has compact support, and let $s_j(k) \stackrel{\text{def}}{=} \langle \phi_{jk}, u\rangle$ and $d_j(k) \stackrel{\text{def}}{=} \langle \psi_{jk}, u\rangle$ be the expansion coefficients of u in V_j and W_J in their respective orthonormal bases. Prove that
$$\begin{aligned} s_{j+1}(n) &= \sum_k h(k)s_j(2n+k) &&= \sum_k h(k-2n)s_j(k); \\ d_{j+1}(n) &= \sum_k g(k)s_j(2n+k) &&= \sum_k g(k-2n)s_j(k); \\ s_{j-1}(n) &= \sum_k \overline{h(n-2k)}s_j(k) + \sum_k \overline{g(n-2k)}d_j(k). \end{aligned}$$
(These are Equations 5.58, 5.59, and 5.60, respectively.)

15. Suppose that x, y, a, b, c, d are integers with $x \geq y$, $b \geq a$, and $d \geq c$.

    a. Show that $2\left\lceil \frac{x-b}{2}\right\rceil + a \leq x-(b-a-1)$, and $2\left\lfloor \frac{y-a}{2}\right\rfloor + b \geq y + (b-a-1)]$. (Hence a sequence u supported in $[x, y]$ may have a projection F^*Fu, as defined by Equation 5.63, with coefficients in the larger interval $[x-(b-a-1), y+(b-a-1)]$.)

    b. Show that
$$\begin{aligned} &\left(1 + \left\lfloor \frac{y-a}{2}\right\rfloor - \left\lceil \frac{x-b}{2}\right\rceil\right) + \left(1 + \left\lfloor \frac{y-c}{2}\right\rfloor - \left\lceil \frac{x-d}{2}\right\rceil\right) \\ &\qquad \geq 1 + y - x + \frac{b-a-1}{2} + \frac{d-c-1}{2}. \end{aligned}$$
    (This estimate of the quantity in Equation 5.67 shows that the nonperiodic discrete wavelet transform with CQFs longer than 2 may have more output coefficients than inputs.)

16. Show that if $h = \{h(k) : k \in \mathbf{Z}\}$ and $g = \{g(k) : k \in \mathbf{Z}\}$ satisfy the orthogonal CQF conditions, and $P = 2P'$ is any fixed *even* integer, then the P-periodizations h_P, g_P of h and g, respectively, also satisfy the orthogonal CQF conditions. Namely, show:

Normalization of h_P: $\sum_{k=0}^{P'-1} h_P(2k) = \sum_{k=0}^{P'-1} h_P(2k+1) = 1/\sqrt{2}$, and thus $\sum_{k=0}^{P-1} h_P(k) = \sqrt{2}$.

Self-orthonormality of h_P: $\sum_{k=0}^{P-1} \overline{h_P(k+2n)} h_P(k+2m) = \delta_{P'}(n-m)$, for all integers n, m.

Normalization of g_P: $\sum_{k=0}^{P'-1} g_P(2k) = -\sum_{k=0}^{P'-1} g_P(2k+1) = 1/\sqrt{2}$, and thus $\sum_{k=0}^{P-1} g_P(k) = 0$.

Self-orthonormality of g_P: $\sum_{k=0}^{P-1} \overline{g_P(k+2n)} g_P(k+2m) = \delta_{P'}(n-m)$, for all integers n, m.

Periodic independence of h_P and g_P: $\sum_{k=0}^{P-1} \overline{g_P(k+2n)} h_P(k+2m) = 0$ for all integers n, m.

Periodic completeness of h_P and g_P: for all integers n, m,

$$\sum_{k=0}^{P'-1} \left[\overline{h_P(2k+n)} h_P(2k+m) + \overline{g_P(2k+n)} g_P(2k+m) \right] = \delta_P(n-m).$$

17. Implement the inverse to Mallat's periodic discrete wavelet transform, for signals of period $N = 2^J K$ with positive integer J and K, using arbitrary 4-tap filters. Use it to generate a graph of the Daubechies 4 wavelet and scaling function, using the filters

$$\begin{aligned}
-g(3) = h(0) &= \frac{1+\sqrt{3}}{4\sqrt{2}} &\approx&\quad 0.48296291314453416 \\
g(2) = h(1) &= \frac{3+\sqrt{3}}{4\sqrt{2}} &\approx&\quad 0.83651630373780794 \\
-g(1) = h(2) &= \frac{3-\sqrt{3}}{4\sqrt{2}} &\approx&\quad 0.22414386804201339 \\
g(0) = h(3) &= \frac{1-\sqrt{3}}{4\sqrt{2}} &\approx&\quad -0.12940952255126037
\end{aligned}$$

18. Suppose that $\{u(n) : n \in \mathbf{Z}, 0 \le n \le 2q-1\}$ is a given sequence of $2q$ numbers.

a. Determine the contents of $u(2n)$ and $u(2n+1)$, for $n = 0, 1, \ldots, q-1$, after the following sequence of substitutions:

1. $u(2n+1) \leftarrow u(2n+1) - u(2n)$, all $n = 0, \ldots, q-1$;
2. $u(2n) \leftarrow u(2n) + \frac{1}{2} u(2n+1)$, all $n = 0, \ldots, q-1$;
3. $u(2n+1) \leftarrow u(2n+1)/\sqrt{2}$, all $n = 0, \ldots, q-1$;
4. $u(2n) \leftarrow \sqrt{2}\, u(2n)$, all $n = 0, \ldots, q-1$.

(These are the steps performed by `haarlift()`.)

b. Determine the contents of $u(2n)$ and $u(2n+1)$, for $n = 0, 1, \ldots, q-1$, after the following sequence of substitutions: