Proof: Properties 1–4 will be shown in turn.

1. We show by induction that ϕ inherits compact support from $\mathbf{1}$. Suppose $h(k) = 0$ for $k > b$ or $k < a$, and $f(t) = 0$ for $t \notin [c, d]$. Then $Hf(t) = \sqrt{2}\sum_{k=a}^{b} h(k)f(2t-k) = 0$ unless $c \le 2t - k \le d$ for some $k = a, a+1, \ldots, b$. But this means that $Hf(t) = 0$ unless $(a+c)/2 \le t \le (b+d)/2$. Iterating, $H^2 f(t) = 0$ unless $[a + (a+c)/2]/2 \le t \le [b + (b+d)/2]/2$, and $H^n f(t) = 0$ unless

$$(1-2^{-n})a+2^{-n}c = \frac{a}{2}+\cdots+\frac{a}{2^n}+\frac{c}{2^n} \le t \le \frac{b}{2}+\cdots+\frac{b}{2^n}+\frac{d}{2^n} = (1-2^{-n})b+2^{-n}d.$$

Note that if $t < -|a| - |c|$ or $t > |b| + |d|$, then $H^n f(t) = 0$ for every $n = 0, 1, 2, \ldots$. This means that u and all iterations $H^n f$ are supported in a single compact interval $[-|a| - |c|, |b| + |d|]$, of length $|a| + |b| + |c| + |d|$. Starting with $\mathbf{1}$, with $c = 0$ and $d = 1$, the limit ϕ obtained as $n \to \infty$ is supported in $[a, b]$.

2. Let $\epsilon > 0$ be given. We show that $|\phi(t) - H\phi(t)| < \epsilon$ for every t, so since ϵ is arbitrary we must have $\phi = H\phi$. But for any $\delta > 0$, by uniform convergence we can choose N large enough so that $n \ge N \Rightarrow |f_n(t) - \phi(t)| < \delta$. Then, using the relation $f_{n+1}(t) = Hf_n(t)$ and the triangle inequality, we estimate

$$|\phi(t) - H\phi(t)| \le |\phi(t) - f_{n+1}(t)| + |Hf_n(t) - H\phi(t)| < \delta + B\delta,$$

where $B = \sqrt{2}\sum_k |h(k)|$ is the largest amount by which H can increase the maximum absolute value of a function. Now we go back and pick $\delta = \epsilon/(1+B)$.

3. First note that if f is integrable then so is Hf, since finite sums of integrable functions are integrable. Then observe that $\int Hf(t)\,dt = \int f(t)\,dt$ by Equation 5.38. Thus, for $n = 0, 1, 2, \ldots$, f_n is integrable and $\int f_n(t)\,dt = 1$. Now $f_n(t) \to \phi(t)$ uniformly at all t, so given $\epsilon > 0$ there is an N large enough so that $n \ge N \Rightarrow |\phi(t) - f_n(t)| < \epsilon$ for all t. Since all functions are supported in a single interval of length $L = |a| + |b| + |c| + |d|$, it follows[4] that ϕ is integrable, and

$$\left|\int \phi(t)\,dt - 1\right| = \left|\int \phi(t)\,dt - \int f_n(t)\,dt\right| \le \int |\phi(t) - f_n(t)|\,dt < \epsilon L.$$

Since $\left|\int \phi(t)\,dt - 1\right| < \epsilon L$ for every $\epsilon > 0$, it follows that $\int \phi(t)\,dt = 1$. Finally, ϕ must be nonzero because its integral is nonzero.

4. Suppose $\langle f(t-p), f(t-q)\rangle = \delta(p-q)$ for a compactly-supported function f. Then Hf inherits the same property:

$$\begin{aligned}\langle Hf(t-p), Hf(t-q)\rangle &= 2\int \sum_{k,k'} \overline{h(k-2p)}h(k'-2q)\overline{f(2t-k)}f(2t-k')\,dt \\ &= \sum_{k,k'} \overline{h(k-2p)}h(k'-2q)\delta(k-k') \;=\; \delta(p-q).\end{aligned}$$

Thus, since $\mathbf{1}$ is orthogonal to its integer translates, so is f_n for every $n = 0, 1, \ldots$.

[4]Here we use the Lebesgue dominated convergence theorem, which is beyond the scope of this text. For a full proof, see Apostol, page 270, in the further readings from Chapter 3.

For n so large that $|\phi(t) - f_n(t)| < \epsilon$ at all t, the value $M = \int |\phi(t)|\, dt$ gives an upper bound $\int |f_n(t)|\, dt < M + \epsilon L$. Thus

$$\begin{aligned}
&\left| \int \overline{\phi(t-p)}\phi(t-q)\, dt \;-\; \delta(p-q) \right| = \\
&\quad = \left| \int \overline{\phi(t-p)}\phi(t-q)\, dt - \int \overline{f_n(t-p)} f_n(t-q)\, dt \right| \\
&\quad \le \int |\phi(t-p) - f_n(t-p)|\, |\phi(t-q)|\, dt \\
&\qquad\qquad + \int |f_n(t-p)|\, |\phi(t-q) - f_n(t-q)|\, dt \\
&\quad < \epsilon M + (M + \epsilon L)\epsilon.
\end{aligned}$$

Since this holds for all $\epsilon > 0$, it follows that $|\langle \phi(t-p), \phi(t-q)\rangle - \delta(p-q)| = 0$. This implies that $\langle \phi(t-p), \phi(t-q)\rangle = \delta(p-q)$, so the inner product is zero if $p \neq q$. Finally, setting $p = q = 0$ shows that $\|\phi\| = 1$. □

For the Haar filters of Equation 5.46, we have $H\mathbf{1} = \mathbf{1}$ as in Equation 5.31. Thus $f_N = \mathbf{1}$ for all $N = 0, 1, 2, \ldots$, so convergence is not only uniform but immediate: $\phi = \mathbf{1}$. However, a study of more general filters H, G that have uniform convergence of $\{f_n\}$ is beyond the scope of this book.[5]

To get lots of samples for a graph of a wavelet or scaling function, however, is easy. Since the wavelet transform of ϕ_{jk} has the expansion sequence $s_j = \mathbf{e}_k$, we simply apply the j-level inverse wavelet transform to the sequences $s_j = \mathbf{e}_k$; $d_j = d_{j-1} = \cdots = d_1 = \mathbf{0}$. Likewise, to get the samples for a graph of ψ_{jk}, we simply apply the j-level inverse wavelet transform to the sequences $s_j = \mathbf{0}$; $d_j = \mathbf{e}_k$; $d_{j-1} = \cdots = d_1 = \mathbf{0}$. We need to use the indexing formula for the particular wavelet transform to find the locations of $s_j(k)$ and $d_j(k)$.

5.2.4 Lifting

A clever method of implementing filter transforms is called *lifting*. The two output sequences (Hu, Gu) produced by a pair H, G of CQFs are computed together, efficiently and in a manner that reduces the amount of auxiliary storage. We illustrate with the example of Haar filters applied to a finitely-supported signal $u(0), \ldots, u(N-1)$ of length $N = 2q > 0$:

Lifting Implementation of the Haar Filter Transform on $2q$ Samples

```
haarlift( u[], q, dq ):
[1] For n=0 to q-1, replace u[(2*n+1)*dq] -=  u[(2*n)*dq]
[2] For n=0 to q-1, replace u[(2*n)*dq] += 0.5*u[(2*n+1)*dq]
[3] For n=0 to q-1, replace u[(2*n+1)*dq] /= sqrt(2.0)
[4] For n=0 to q-1, replace u[(2*n)*dq] *= sqrt(2.0)
```

[5] Cavaretta, Dahmen and Micchelli's *Stationary Subdivision*, in the further readings, has a highly detailed exposition of the relevant technicalities.