Thus $\psi(t-k) \in V_{-1}$ for every integer k. In fact, $\{\psi(t-k) : k \in \mathbf{Z}\}$ is an orthonormal subset of V_{-1}:

$$\begin{aligned}
\langle \psi(t-n), \psi(t-m) \rangle &= \\
&= \sum_l \sum_k \overline{g(l)} g(k) \left\langle \sqrt{2}\,\phi(2t-2n-l), \sqrt{2}\,\phi(2t-2m-k) \right\rangle \\
&= \sum_l \sum_k \overline{g(l)} g(k) \delta(2n+l-2m-k) \\
&= \sum_k \overline{g(2(m-n)+k)} g(k) \qquad = \delta(n-m).
\end{aligned}$$

This mother function defines another collection of subspaces in the MRA. Put $W_0 = \operatorname{span}\{\psi(t-k) : k \in \mathbf{Z}\}$, and observe that $W_0 \subset V_{-1}$. In general, for any integer j, put

$$W_j \stackrel{\text{def}}{=} \operatorname{span}\{\psi(2^{-j}t-k) : k \in \mathbf{Z}\}. \tag{5.51}$$

Then $W_j \subset V_{j-1}$. Note that $\{2^{-j/2}\psi(2^{-j}t-k) : k \in \mathbf{Z}\}$ is an orthonormal basis for W_j.

By the independence condition, every basis vector of W_0 is orthogonal to every basis vector of V_0:

$$\begin{aligned}
\langle \psi(t-n), \phi(t-m) \rangle &= \\
&= \sum_l \sum_k \overline{g(l)} h(k) \left\langle \sqrt{2}\,\phi(2t-2n-l), \sqrt{2}\,\phi(2t-2m-k) \right\rangle \\
&= \sum_l \sum_k \overline{g(l)} h(k) \delta(2n+l-2m-k) \\
&= \sum_k \overline{g(2(n-m)+k)} h(k) \quad = 0.
\end{aligned}$$

Also, since $\langle \psi(t-n), \phi(t-m) \rangle = 2^j \left\langle \psi(2^j t-n), \phi(2^j t-m) \right\rangle$ for every $j \in \mathbf{Z}$, every basis vector of W_j is orthogonal to every basis vector of V_j. In other words, $W_j \perp V_j$.

Multiresolution analysis works because $f \in V_{-1}$ is the sum of an average part that lies in V_0 and a complementary detail part that lies in W_0:

Lemma 5.8 $W_0 + V_0 = V_{-1}$.

Proof: We first show that each basis function of V_{-1} is a sum of a function in V_0 and a function in W_0, namely, that

$$\sqrt{2}\phi(2t-n) = \sum_k \overline{h(n-2k)}\phi(t-k) + \sum_k \overline{g(n-2k)}\psi(t-k). \tag{5.52}$$

Using the two-scale relations for the scaling and mother functions, we may expand the ϕ and ψ terms. Then we use Equation 5.45, the completeness condition, to evaluate the sum over index k as follows:

$$\begin{aligned}\sum_k \overline{h(n-2k)}\phi(t-k) + \sum_k \overline{g(n-2k)}\psi(t-k) &= \\ &= \sum_k \sum_m \overline{h(n-2k)}h(m)\sqrt{2}\phi(2t-2k-m) \\ &\qquad + \sum_k \sum_m \overline{g(n-2k)}g(m)\sqrt{2}\phi(2t-2k-m) \\ &= \sum_m \left(\sum_k \overline{h(n{-}2k)}h(m{-}2k) + \overline{g(n{-}2k)}g(m{-}2k) \right) \sqrt{2}\phi(2t-m) \\ &= \sum_m \delta(n-m)\sqrt{2}\phi(2t-m) \quad = \quad \sqrt{2}\phi(2t-n).\end{aligned}$$

Thus, for any $u = u(t) = \sum_k c(k)\sqrt{2}\phi(2t-k) \in V_{-1}$, there is a function $P_0 u(t) \stackrel{\text{def}}{=} \sum_k s(k)\phi(t-k) \in V_0$, where $s(k) = \langle \phi(t-k), u(t) \rangle$, and a function $Q_0 u(t) \stackrel{\text{def}}{=} \sum_k d(k)\psi(t-k) \in W_0$, where $d(k) = \langle \psi(t-k), u(t) \rangle$, and since $c(n) = \sum_k \overline{h(n-2k)}s(k) + \sum_k \overline{g(n-2k)}d(k)$, it follows that $u = P_0 u + Q_0 u$. □

This decomposition generalizes to arbitrary scales in the MRA. For fixed $j \in \mathbf{Z}$, define the functions

$$\phi_{jk}(t) \stackrel{\text{def}}{=} 2^{-j/2}\phi(2^{-j}t-k), \qquad k \in \mathbf{Z}, t \in \mathbf{R} \tag{5.53}$$

$$\psi_{jk}(t) \stackrel{\text{def}}{=} 2^{-j/2}\psi(2^{-j}t-k), \qquad k \in \mathbf{Z}, t \in \mathbf{R} \tag{5.54}$$

These are orthonormal basis vectors for V_j and W_j, respectively.

Corollary 5.9 *For every integer j, $W_j + V_j = V_{j-1}$.*

Proof: We substitute $t \leftarrow 2^{-j}t$ and multiply by $2^{-j/2}$ everywhere in Equation 5.52 in the proof of Lemma 5.8, then apply the definitions of ϕ_{jk} and ψ_{jk} to get

$$\phi_{j-1,n}(t) = \sum_k \overline{h(n-2k)}\, \phi_{jk}(t) + \sum_k \overline{g(n-2k)}\, \psi_{jk}(t). \tag{5.55}$$

We have thus written an arbitrary basis function of V_{j-1} as a linear combination of basis functions of V_j and W_j. □

The subspaces W_j are the differences between the adjacent V_j and V_{j-1}. Knowing the expansion coefficients of u's approximation in V_j, it is only necessary to get the expansion coefficients of its projection on W_j (and to do some arithmetic) in order to get a better approximation of u in V_{j-1}. We may call W_j a *detail* space, since it contains the details from u's approximation in V_{j-1} which are missing in V_j. Repeated application of this splitting yields the *discrete wavelet decomposition*:

Corollary 5.10 *$V_0 = W_1 + W_2 + \cdots + W_J + V_J$, for any integer $J > 0$.* □

If the scale and detail spaces form an orthogonal MRA, then the subspaces in the sum are pairwise orthogonal.