Third, we observe that another filter with similar orthogonality properties can be defined from h:

$$g(k) = (-1)^k \overline{h(1-k)}, \qquad \text{for all } k \in \mathbf{Z}. \tag{5.41}$$

Clearly, g will be finite whenever h is finite, and given g we may determine h by the similar formula $h(k) = (-1)^{1-k}\overline{g(1-k)}$. This and Equation 5.40 implies the *high-pass filter condition* for g:

$$\sum_k g(2k) = -\sum_k g(2k+1) = \frac{1}{\sqrt{2}} \qquad \left(\Rightarrow \sum_k g(k) = 0\right). \tag{5.42}$$

Fourth, there is a *self-orthonormality condition* for g:

$$\begin{aligned} \sum_k \overline{g(k)}g(k+2n) &= \sum_k (-1)^k h(1-k)(-1)^{k+2n}\overline{h(1-k-2n)} \\ &= \sum_k h(1-k)\overline{h(1-k-2n)} \\ &= \sum_k h(k)\overline{h(k-2n)} \quad = \delta(n). \end{aligned} \tag{5.43}$$

Fifth, for any integer n, the following *independence condition* holds between the two filters h and g:

$$\begin{aligned} \sum_k \overline{g(k)}h(k+2n) &= \sum_k (-1)^k h(1-k)h(k+2n) \qquad (5.44) \\ &= \sum_{\text{even } k} h(1-k)h(k+2n) - \sum_{\text{odd } k} h(1-k)h(k+2n) \\ &= \sum_p h(2p{+}1)h(2n{-}2p) - \sum_q h(2n{-}2q)h(2q{+}1) \ = \ 0. \end{aligned}$$

Here $k \leftarrow -2p$ in the first sum and $k \leftarrow 2q+1-2n$ in the second.

Finally, the filter pair h, g satisfies the *completeness* condition:

$$\sum_k \overline{h(2k+n)}h(2k+m) + \sum_k \overline{g(2k+n)}g(2k+m) = \delta(n-m). \tag{5.45}$$

This can be shown case-by-case. We first write g in terms of h, making the sum

$$\sum_k \overline{h(2k+n)}h(2k+m) + (-1)^{n+m}\sum_k h(2k+1-n)\overline{h(2k+1-m)}.$$

Then we put $p = m-n$ to have $n+m = 2n+p$ and $(-1)^{n+m} = (-1)^p$, and consider the cases:

- If $n = 2n'$ is even, then substituting $k \leftarrow k-n'$ in the first sum and $k \leftarrow k+n'$ in the second reduces them to

$$\sum_k \overline{h(2k)}h(2k+p) + (-1)^p \sum_k h(2k+1)\overline{h(2k+1-p)}.$$

– If $p = 2p'$ is even, then substituting $k \leftarrow k + p'$ in the second sum gives

$$\begin{aligned}\sum_k \overline{h(2k)}h(2k{+}p) + \sum_k h(2k{+}1{+}p)\overline{h(2k+1)} &= \sum_k \overline{h(k)}h(k{+}p) \\ = \sum_k \overline{h(k)}h(k+2p') &= \delta(p') \;=\; \delta(n{-}m).\end{aligned}$$

– If $p = 2p' + 1$ is odd, then substituting $k \leftarrow k + p'$ in the second sum gives

$$\sum_k \overline{h(2k)}h(2k+p) - \sum_k h(2k+p)\overline{h(2k)} = 0.$$

This agrees with the value of $\delta(n - m)$, which is 0 in this case since $p = m - n$ being odd means $n \neq m$.

- If $n = 2n' + 1$ is odd, then substituting $k \leftarrow k - n'$ in the first sum and $k \leftarrow k + n'$ in the second reduces them to

$$\sum_k \overline{h(2k+1)}h(2k+1+p) + (-1)^p \sum_k h(2k)\overline{h(2k-p)}.$$

– If $p = 2p'$ is even, then substituting $k \leftarrow k + p'$ in the second sum gives

$$\begin{aligned}\sum_k \overline{h(2k{+}1)}h(2k{+}1{+}p) + \sum_k h(2k+p)\overline{h(2k)} &= \sum_k \overline{h(k)}h(k+p) \\ = \sum_k \overline{h(k)}h(k+2p') &= \delta(p') \;=\; \delta(n{-}m).\end{aligned}$$

– If $p = 2p' - 1$ is odd, then substituting $k \leftarrow k + p'$ in the second sum gives

$$\sum_k \overline{h(2k+1)}h(2k+1+p) - \sum_k h(2k+p+1)\overline{h(2k+1)} = 0.$$

This agrees with the value of $\delta(n - m)$, which is 0 in this case since $p = m - n$ being odd means $n \neq m$.

The sequences h and g derived from the MRA are called *orthogonal conjugate quadrature filters*, or orthogonal CQFs. We may abstract the properties just deduced from the MRA conditions:

Orthogonal CQF Conditions (Basic)

Finiteness: Sequence $h = \{h(k) : k \in \mathbf{Z}\}$ consists of zeroes for all but finitely many values of k.

Normalization of h: $\sum_k h(2k) = \sum_k h(2k+1) = 1/\sqrt{2}$, and thus $\sum_k h(k) = \sqrt{2}$.

Self-Orthonormality of h: $\sum_k \overline{h(k+2n)}h(k+2m) = \delta(n-m)$, for every $n, m \in \mathbf{Z}$.

From these stand-alone assumptions, the other properties of h and g can be deduced:

Orthogonal CQF Conditions (Derived)

Conjugacy: For some fixed integer M there is a finitely-supported sequence $g = \{g(k) : k \in \mathbf{Z}\}$, defined by $g(k) = (-1)^k \overline{h(2M-1-k)}$ for each $k \in \mathbf{Z}$.

Normalization of g: $\sum_k g(2k) = -\sum_k g(2k+1) = 1/\sqrt{2}$, and thus $\sum_k g(k) = 0$.

Self-Orthonormality of g: $\sum_k \overline{g(k+2n)} g(k+2m) = \delta(n-m)$.

Independence: $\sum_k \overline{g(k+2n)} h(k+2m) = 0$ for any $n, m \in \mathbf{Z}$.

Completeness: $\sum_k \overline{h(2k+n)} h(2k+m) + \sum_k \overline{g(2k+n)} g(2k+m) = \delta(n-m)$.

The so-called *lazy filters*, $h(k) = \sqrt{2}\,\delta(k-1)$ and $g(k) = \sqrt{2}\,\delta(k)$ satisfy the finiteness, conjugacy, self-orthonormality, independence and completeness conditions, but only part of the normalization conditions. This partial example is a useful test case for some constructions.

To be definite, suppose that for some fixed $L > 0$, $h(k) = 0$ if $k < 0$ or $k \geq L$; this may be called *conventional indexing*. Then the length of the finite support of h is no more than L. If it is exactly L, namely if $h(0) \neq 0$ and $h(L-1) \neq 0$, then h is said to have *filter length L*. The normalization condition implies that filter length L is at least two. Orthogonality imposes an additional constraint:

Lemma 5.7 *An orthogonal conjugate quadrature filter's length must be even.*

Proof: It is enough to prove this for the low-pass filter h, since the high-pass conjugate filter g will have the same length L as h. If $L = 2l+1$ for $l > 0$, then $L - 1 = 2l$ is the largest index k for which $h(k) \neq 0$, so

$$0 = \sum_k \overline{h(k)} h(k+2l) = \overline{h(0)} h(2l) = h(0)h(L-1).$$

Thus either $h(0) = 0$ or $h(L-1) = 0$, contradicting the assumption that h has length L. □

Constructing orthogonal filter pairs

How can we construct a finite sequence $h = \{h(k) : k \in \mathbf{Z}\}$ satisfying the orthogonal CQF conditions?

One solution can be found right away, the *Haar filter*, which is the unique orthogonal CQF of length two:

$$h(k) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } k = 0 \text{ or } k = 1, \\ 0, & \text{if } k \notin \{0,1\}; \end{cases} \qquad g(k) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } k = 0, \\ -\frac{1}{\sqrt{2}}, & \text{if } k = 1, \\ 0, & \text{if } k \notin \{0,1\}. \end{cases} \tag{5.46}$$