ξ. Thus, it is legal to differentiate once or twice with respect to a or b under the integral sign:

$$\begin{aligned}\frac{\partial}{\partial a}Wu(a,b) &= \int_{-\infty}^{\infty} \xi e^{-2\pi i b\xi}\phi'(a\xi)\mathcal{F}u(\xi)\,d\xi;\\ \frac{\partial}{\partial b}Wu(a,b) &= -2\pi i\int_{-\infty}^{\infty} \xi e^{-2\pi i b\xi}\phi(a\xi)\mathcal{F}u(\xi)\,d\xi.\end{aligned}$$

It remains to show that these derivatives are continuous functions of a, b away from the line $a = 0$. But in both cases, this follows from the observation that the integrands are continuous functions of a, b. □

8. **Solution:** Since $w = \mathcal{F}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$, use Plancherel's theorem to compute $\|w\| = \|\mathcal{F}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}\| = \|\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}\| = 1$. □

9. **Solution:** By the previous solution and by combining integrals, calculate that $\mathcal{F}w = \mathbf{1}_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]}$. Thus,

$$c_w = \int_0^{\infty} \frac{|\mathbf{1}_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]}(\xi)|^2}{\xi}\,d\xi = \int_{\frac{1}{2}}^{1}\frac{d\xi}{\xi} = \log 2 \approx 0.69315 < \infty.$$

But $\mathcal{F}w(-\xi) = \mathcal{F}w(\xi)$, so the $-\xi$ integral is the same, so w is admissible. □

10. **Solution:** The Fourier integral transform of w is

$$\int_{-\infty}^{\infty} e^{-2\pi i x\xi} w(x)\,dx.$$

Since $w(x) = 1$ if $0 < x < \frac{1}{2}$ and $w(x) = -1$ if $\frac{1}{2} < x < 1$, that simplifies to

$$\int_0^{\frac{1}{2}} e^{-2\pi i x\xi} - \int_{\frac{1}{2}}^{1} e^{-2\pi i x\xi} = \frac{\left(e^{-\pi i\xi} - 1\right)^2}{2\pi i\xi}.$$

□

11. **Solution:** It is necessary to show that $\langle\phi_j, \phi_k\rangle = \delta(j-k)$. But Plancherel's theorem allows writing

$$\langle\phi_j, \phi_k\rangle = \langle\mathcal{F}\phi_j, \mathcal{F}\phi_k\rangle = \int_{-1/2}^{1/2} e^{2\pi i(k-j)\xi}\,d\xi = \delta(j-k),$$

since $\mathcal{F}\phi_k(\xi) = e^{2\pi i k\xi}\mathcal{F}\mathrm{sinc}\,(\xi) = e^{2\pi i k\xi}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$. □

12. **Solution:** Show that $\sum_k g(2k) = -\sum_k g(2k+1) = \frac{1}{\sqrt{2}}$:

$$\begin{aligned}\sum_k g(2k) &= \sum_k (-1)^{2k}\overline{h(2M-1-2k)} = \sum_k \overline{h(2M-1-2k)}\\ &= \sum_k \overline{h(2(M-k)-1)} = \sum_l \overline{h(2l+1)} = \frac{1}{\sqrt{2}},\end{aligned}$$

after the substitution $l \leftarrow M - k$. A similar change of the index gives:

$$\begin{aligned}\sum_k g(2k+1) &= \sum_k (-1)^{2k+1}\overline{h(2M-1-(2k+1))} \\ &= -\sum_k \overline{h(2(M-k-1))} = -\sum_l \overline{h(2l)} \quad = -\frac{1}{\sqrt{2}}.\end{aligned}$$

Complex conjugation has no effect on the sums, as $1/\sqrt{2}$ is purely real. □

13. **Solution:** First note that $1 + c^2 = 8 - 4\sqrt{3} = 4c$, so $c/(1 + c^2) = \frac{1}{4}$. Thus

$$h(2) = \frac{c(c+1)}{\sqrt{2}(1+c^2)} = \frac{c+1}{4\sqrt{2}} = \frac{(2-\sqrt{3})+1}{4\sqrt{2}} = \frac{3-\sqrt{3}}{4\sqrt{2}}.$$

Likewise,

$$h(3) = \frac{c(c-1)}{\sqrt{2}(1+c^2)} = \frac{c-1}{4\sqrt{2}} = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

Finally,

$$h(0) = \frac{1}{\sqrt{2}} - h(2) = \frac{1+\sqrt{3}}{4\sqrt{2}}; \qquad h(1) = \frac{1}{\sqrt{2}} - h(3) = \frac{3+\sqrt{3}}{4\sqrt{2}}.$$

□

14. **Solution:** We begin by substituting $t \leftarrow 2^{-j-1}t$ and then multiplying by $2^{-(j+1)/2}$ on both sides of Equation 5.33:

$$\begin{aligned}2^{-(j+1)/2}\phi(2^{-j-1}t - n) &= \sum_k h(k)2^{-j/2}\,\phi(2^{-j}t - 2n - k) \\ &= \sum_{k'} h(k' - 2n)2^{-j/2}\,\phi(2^{-j}t - k').\end{aligned}$$

Taking inner products on both sides with u gives the first result.

Similarly, we may substitute $t \leftarrow 2^{-j-1}t - n$ and then multiply by $2^{-(j+1)/2}$ in Equation 5.50 to get

$$\begin{aligned}2^{-(j+1)/2}\psi(2^{-j-1}t - n) &= \sum_k g(k)2^{-j/2}\,\phi(2^{-j}t - 2n - k) \\ &= \sum_{k'} g(k' - 2n)2^{-j/2}\,\phi(2^{-j}t - k').\end{aligned}$$

Taking inner products on both sides with u gives the second result.

For the third result, we take inner products with u on both sides of Equation 5.55. □