Filters of length four are not unique. Let h be an orthogonal CQF with nonzero real coefficients $h(0)$, $h(1)$, $h(2)$, and $h(3)$. Then h must satisfy the norm condition $h^2(0) + h^2(1) + h^2(2) + h^2(3) = 1$, plus the following constraints:

$$h(0) + h(2) = \frac{1}{\sqrt{2}}; \qquad h(1) + h(3) = \frac{1}{\sqrt{2}}; \qquad h(0)h(2) + h(1)h(3) = 0. \quad (5.47)$$

By the first two conditions, picking $h(0)$ and $h(1)$ determines $h(2) = \frac{1}{\sqrt{2}} - h(0)$ and $h(3) = \frac{1}{\sqrt{2}} - h(1)$. The third condition holds if and only if there is some real number c for which $h(2) = ch(1)$ and $h(3) = -ch(0)$. The result is a system of two linear equations for $h(0)$ and $h(1)$, containing a free parameter c:

$$\begin{array}{rcl} h(0) + ch(1) & = & \frac{1}{\sqrt{2}} \\ -ch(0) + h(1) & = & \frac{1}{\sqrt{2}} \end{array} \quad \Rightarrow \quad \begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix} \begin{pmatrix} h(0) \\ h(1) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The matrix is nonsingular for every real c since its determinant, $1 + c^2$, is at least one, and the one-parameter set of solutions is obtainable by inverting:

$$\begin{pmatrix} h(0) \\ h(1) \end{pmatrix} = \frac{1}{\sqrt{2}(1+c^2)} \begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} h(0) & = & \frac{1-c}{\sqrt{2}(1+c^2)} \\ h(1) & = & \frac{1+c}{\sqrt{2}(1+c^2)} \end{array} \quad (5.48)$$

The remaining coefficients are then $h(2) = \frac{c(c+1)}{\sqrt{2}(1+c^2)}$ and $h(3) = \frac{c(c-1)}{\sqrt{2}(1+c^2)}$.

The *Daubechies 4 filters* are obtained this way using $c = 2 - \sqrt{3}$:

$$h(0) = \frac{1+\sqrt{3}}{4\sqrt{2}}; \quad h(1) = \frac{3+\sqrt{3}}{4\sqrt{2}}; \quad h(2) = \frac{3-\sqrt{3}}{4\sqrt{2}}; \quad h(3) = \frac{1-\sqrt{3}}{4\sqrt{2}}. \quad (5.49)$$

The normalization condition for 5.47 seems to impose an additional constraint on c. However, that condition is satisfied for all real c:

$$\begin{aligned} h^2(0)+h^2(1)+h^2(2)+h^2(3) &= (1+c^2)(h^2(0) + h^2(1)) \\ &= (1+c^2)\left(\frac{1-2c+c^2}{2(1+c^2)^2} + \frac{1+2c+c^2}{2(1+c^2)^2}\right) = 1. \end{aligned}$$

If all four coefficients are nonzero, then $c \notin \{0, \pm 1, \pm\infty\}$. Otherwise, the degenerate cases are

$$\begin{array}{lll} c = -1 \Rightarrow h = \{\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\}; & \qquad & c = 0 \Rightarrow h = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\}; \\ c = 1 \Rightarrow h = \{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\}; & \qquad & c = \pm\infty \Rightarrow h = \{0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}. \end{array}$$

These are all just variations on the Haar filters.

Mother functions and details

The conjugate filter g derived from h defines the *mother function* for the MRA by way of a linear transformation G:

$$\psi(t) = \sum_k g(k)\sqrt{2}\,\phi(2t-k) \stackrel{\text{def}}{=} G\phi(t). \quad (5.50)$$