and a function f belongs to $L^2(\mathbf{R})$ if and only if $\|f\|$ is finite. Thus the nonzero constant functions, and more generally the nonzero polynomials, are not members of $L^2(\mathbf{R})$. Since $\|f\|^2$ is called the *energy* of a function, L^2 is sometimes called the space of finite-energy signals.

L^2, like **Lip**, is infinite dimensional. Since there is no continuity assumption, we may build a simple set of basis functions from the *indicator function* of the unit interval $[0, 1)$:

$$\mathbf{1}(t) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } 0 \le t < 1; \\ 0, & \text{if } t < 0 \text{ or } t \ge 1. \end{cases} \tag{2.25}$$

Given any integer k, put $e_k(t) \stackrel{\text{def}}{=} \mathbf{1}(t-k)$ to get the characteristic function of $[k, k+1)$. The functions $\{e_k : k \in \mathbf{Z}\}$ are clearly linearly independent in $L^2(\mathbf{R})$, and there are infinitely many of them. We can also introduce a *scale* index j and put $e_{jk}(t) \stackrel{\text{def}}{=} 2^{-j/2}\mathbf{1}(2^{-j}t-k)$, which is normalized to guarantee $\|e_{jk}\| = 1$. The set $\{e_{jk} : j, k \in \mathbf{Z}\}$ is dense in L^2, but it is clearly not linearly independent. However, the fixed-scale functions $E_j = \{e_{jk} : k \in \mathbf{Z}\}$ are linearly independent, and given a function $f \in L^2(\mathbf{R})$ and $\epsilon > 0$, we can find a scale J and a function $f_J \in \operatorname{span} E_J \subset L^2(\mathbf{R})$ satisfying $\|f - f_J\| < \epsilon$.

2.1.3 Inner product spaces

An *inner product space* $\mathbf{X}$ is a special kind of vector space in which there is also an *inner product*. This is a scalar-valued function on pairs of vectors $\mathbf{u}, \mathbf{v} \in \mathbf{X}$, denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$, that must satisfy the following:

Inner Product Axioms

Hermitean symmetry: For any $\mathbf{u}, \mathbf{v} \in \mathbf{X}$, $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

Positive definiteness: If $\mathbf{u} \in \mathbf{X}$ and $\mathbf{u} \neq \mathbf{0}$, then $\langle \mathbf{u}, \mathbf{u} \rangle > 0$.

Linearity: For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$ and any scalars c, d, $\langle \mathbf{u}, c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle + d\langle \mathbf{u}, \mathbf{w} \rangle$.

Hermitean symmetry implies that $\langle \mathbf{u}, \mathbf{u} \rangle$ is purely real. If all coordinates and scalars are real numbers, it reduces to the ordinary symmetry condition $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

Positive definiteness implies *nondegeneracy of the inner product*: $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in \mathbf{X}$ only if $\mathbf{u} = \mathbf{0}$. It also allows us to define a nondegenerate *derived norm* by the formula $\|\mathbf{u}\| \stackrel{\text{def}}{=} \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \ge 0$, just as in Euclidean N-space. Linearity implies that $\|\mathbf{0}\|^2 = \langle \mathbf{0}, \mathbf{0} \rangle = 0$, so we have $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

By linearity and Hermitean symmetry, $\langle c\mathbf{v} + d\mathbf{w}, \mathbf{u} \rangle = \bar{c}\langle \mathbf{v}, \mathbf{u} \rangle + \bar{d}\langle \mathbf{w}, \mathbf{u} \rangle$. Thus $\langle c\mathbf{u}, c\mathbf{u} \rangle = |c|^2 \langle \mathbf{u}, \mathbf{u} \rangle$, so the derived norm satisfies $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$. We will see in Lemma 2.4 that the other sublinearity condition also holds, so a derived norm indeed satisfies the norm axioms.

If all scalars and coordinates are real numbers, the inner product is real-valued and linear in the first factor as well: $\langle c\mathbf{v} + d\mathbf{w}, \mathbf{u} \rangle = c\langle \mathbf{v}, \mathbf{u} \rangle + d\langle \mathbf{w}, \mathbf{u} \rangle$.