EXAM I
Math 109 / Music 109A, Spring 2009

Name ___________ Id ___________

Each problem is worth 10 points.

1. Aural: Notate the rhythm (one measure each).

   (a) \[ \frac{\hat{\text{d}} \text{ d} \text{ d}}{\text{d}} \]  
   (b) \[ \frac{\text{d} \text{ d} \text{ d}}{\text{d} \text{ d} \text{ d}} \]

   Circle the triad type.

   (c) \underline{\text{major}}  \quad \underline{\text{D}}  \quad \underline{\text{minor}}  \quad \underline{\text{diminished}}

2. Sketch the graphs of these functions by starting with a more basic function and applying one or more geometric transformations (shifts or stretches). Use the space on page 4 if you need it.

   (a) \( f(x) = \frac{1}{2}x^2 \)  
   (b) \( g(x) = -1 + \sin 2x \)

3. For the following pairs of integers \( m, n \), find the numbers \( q \) and \( r \) whose existence is asserted in the division algorithm \( n = qm + r \):

   (a) 11, -101 \quad -101 = -10 \cdot 11 + 9 \quad q = -10 \quad r = 9

   (b) 5, 3035\ell + 9, where \( \ell \) some integer.  
      \[ 3035\ell + 9 = (607\ell + 1)5 + 4 \]
      \[ q = 607\ell + 1, \quad r = 4 \]
4. Write the indicated note as a whole note on the given staff, choosing and notating an appropriate clef.

(a) \[ \text{G}_2 \]  \hspace{1cm} (b) \[ \text{C}_5 \]  \hspace{1cm} (c) \[ \text{B}_3 \]

5. For the set \( \{(a,b) \in \mathbb{Z}^2 \mid b \neq 0\} \) show that the relation \( \sim \) defined by \( (a,b) \sim (a',b') \) if and only if \( ab' - a'b = 0 \) is an equivalence relation. Explain how the set of equivalence classes are in one-to-one correspondence with the set of rational numbers \( \mathbb{Q} \).

Or

For the set \( \mathbb{Z} \) and a fixed positive integer \( m \), show that the relation \( \equiv \) defined by \( k \equiv \ell \) if and only if \( m| (k - \ell) \) is an equivalence relation. Explain why there are exactly \( m \) equivalence classes.

**First:** \( (a,b) \sim (a,b) \), since \( ab - ab = 0 \), so \( \sim \) is reflexive.

If \( (a,b) \sim (a',b') \) then \( ab' - a'b = 0 \), so \( ab' - a'b = 0 \) (multiplying by -1), so \( (a',b') \sim (a,b) \). Thus \( \sim \) is symmetric.

If \( (a,b) \sim (a',b') \) and \( (a',b') \sim (a'',b'') \), then \( ab' - a'b = 0 \) and \( a'b'' - a''b = 0 \). Want to show \( a'b'' - a''b = 0 \).

Multiply \( (a'b'' - a''b) + (a'b'' - a''b) = 0 \)

Define \( \phi : \mathbb{Z} \rightarrow \mathbb{Q} \) by \( (a,b) \mapsto \frac{a}{b} \). This is well-defined, since if \( (a,b) \sim (a',b') \) we have \( \frac{a}{b} = \frac{a'}{b'} \) which is exactly the same as saying \( ab' - a'b = 0 \). This \( \phi \) is one-to-one. Clearly every \( \frac{a}{b} \in \mathbb{Q} \) is in the image, so \( \phi \) is onto as well.

**Second:** \( m \mid (k - \ell) = 0 \), so \( \equiv \) is reflexive. If \( k \equiv l \) and \( l \equiv h \), then \( h - l = am \), \( l - h = bm \). So \( k - l = (a + b)m \). So \( m \mid (k - h) \) as \( k \equiv h \), so \( \equiv \) is transitive.

If \( k \equiv l \), then \( a = h - l \), so \( m - l = -k \) shows \( l \equiv h \). So \( \equiv \) is symmetric.

The classes \( \mathbb{Z} \) are distinct, since any two of \( \mathbb{Z} \) differ by less than \( m \). For any \( n \in \mathbb{Z} \), write \( n = qm + r \) using Division algorithm. Then \( n \equiv r \) and \( r \in \{0, \ldots, m-1\} \).
6. For the following modes and tonic notes, indicate the appropriate key signature on the given staff, taking note of the clef:

(a) Lydian with tonic D

(c) Aeolian with tonic B♭

7. Add the needed sharps or flats to notes so that the following gives the Locrian scale tones 1 to 8, from D to D. (Do not alter D.)

8. Extend the following melody with two measures having the same rhythm, employing the following transformations. Do not write in a key change.

(a) diatonic up one scale tone in the second measure

(b) chromatic up a major third (from the original) in the third measure

9. Give the total duration in beats of:

(a) a doubly-dotted half note in \( \frac{4}{4} \) time. \( d = 2 \quad d'' = 2 \cdot \left(1 + \frac{1}{2} + \frac{1}{4}\right) = 2 \cdot \frac{7}{4} = \frac{7}{2} \)

(b) a half note in \( \frac{6}{8} \) time (compound time signature). \( \frac{1}{4} \quad d' = \frac{1}{4} \quad d'' = \frac{4}{3} \)

(c) a sixteenth note 9-tuplet in \( \frac{4}{4} \) time.

\[ \begin{align*}
9 \quad \frac{1}{2}^\circ & 4 = n + r \\
\frac{2}{3} \leq n \leq 16 & \quad \text{so } r = 3 \\
\text{So } n = 4 & \quad \text{Duration is that } 9 \cdot \frac{1}{2}' \text{-note } = d', \text{ which is } 2 \text{ beats}.
\end{align*} \]
10. For the song *Mary Had A Little Lamb*, give the form (e.g., AABC) by dividing it into segments consisting of two bars. Locate and identify a translation other than that which comes from the overall form.

Mary had a little lamb, little lamb, little lamb,

Mary had a little lamb, his fleece was white as snow.

\[ A \quad B \quad A \quad C \]

*Phrasal translation:* m 2 $\rightarrow$ 3 $\rightarrow$ 4

*Harmonic transposition:* m 2 $\rightarrow$ 3