EXAM II

Math 109 / Music 109A, Spring 2018

Name Solutions Id ____________

Each problem is worth 10 points. Round off each decimal approximation to two digits to the right of the decimal.

1. Express each of these musical intervals as an element of $\mathbb{R}^+$ three ways: (1) as a power of 2, (2) as a radical or the reciprical of a radical, and (3) by a decimal approximation.

   (a) down 46 cents $2^{-46/1200} = \frac{1}{\sqrt[1200]{2^{46}}} \approx 0.97$

   (b) up a minor sixth $2^{8/12} = \sqrt[12]{2^8} \approx 1.59$

2. Convert to the specified additive measurement the intervals given by the following ratios.

   (a) 19/16, convert to semitones $12 \log_2 \frac{19}{16} \approx 2.98$

   (b) $\pi/6$, convert to cents $1200 \log_2 \frac{\pi}{6} \approx -1120.16$

3. A string on a stringed instrument has length 50 cm. Indicate the positions of the single fret which will allow the string to play the note (a) a keyboard minor third above the original pitch, and (b) a ratio 6/5 with the original pitch.

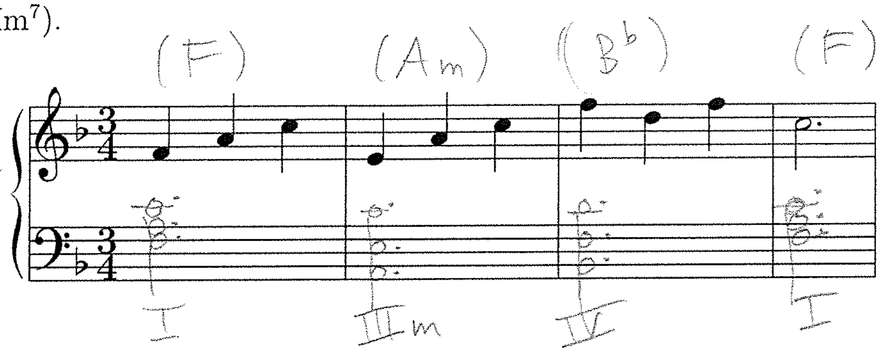
$r = \frac{F'}{F} = \frac{L}{L'} = \frac{50}{L'}$ so $L' = 50r^{-1}$

(a) $r = 2^{3/12} = 2^{1/4}$ $L' = 50 \cdot 2^{-1/4} \approx 42.04$ cm

(b) $r = \frac{6}{5}$ $L' = 50 \cdot \frac{5}{6} \approx 41.67$ cm

4. Give a plausible harmonization of this melody in F major by providing, in the bass clef, one dotted half note chord for each measure. The chords should accommodate every melody note – no non-chord tones. Label each chord by root scale tone (Roman numeral) and chord type (e.g., IIm^7).

For example:



There are other correct choices.

5. Evaluate these logarithms <u>without a calculator</u>. Write down each step of the simplification. You may express your answer as a fraction.

(a) $\log_3\left(\dfrac{81}{\sqrt{3}}\right) = \log_3 81 - \log_3 \sqrt{3} = \log_3 3^4 - \log_3 3^{1/2}$
$= 4 - \frac{1}{2} = \frac{7}{2}$

(b) $\log_b\left(\dfrac{b^p}{\sqrt[m]{b^\ell}}\right) = \log_b (b^p) - \log_b \sqrt[m]{b^\ell} = p\log_b b - \log_b b^{\ell/m}$
$= p - \frac{\ell}{m} = \frac{pm - \ell}{m}$

6. Write on the staff the keyboard note which best approximates the frequency having the given interval ratio $r = 3/13$ from the given note. Compute the error in cents.

$12 \log_2 \frac{3}{13} \approx -25.39$

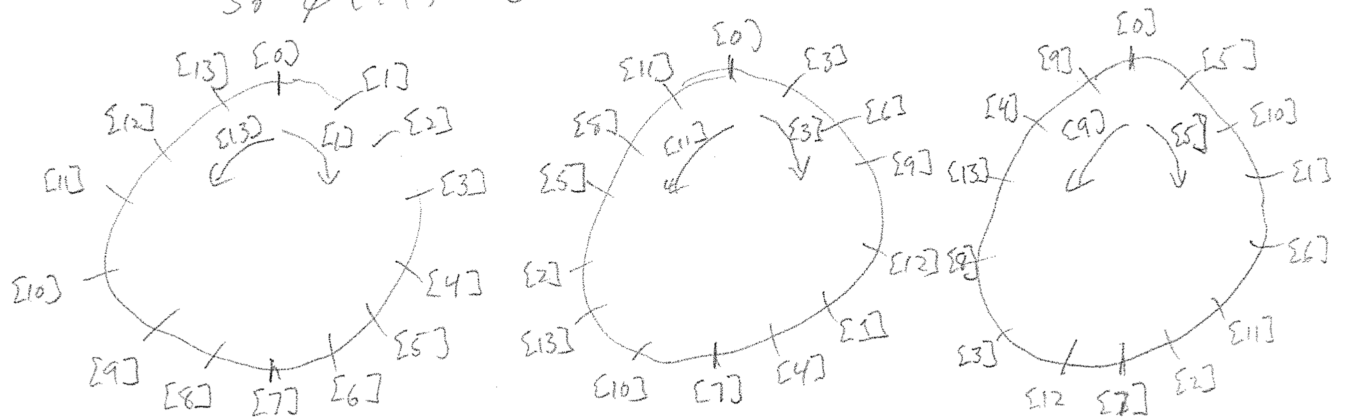
rounds to -25 semitones below G_5 which is $F^\#_3$.

This is 39 cents above the desired note

7. Determine $\phi(14)$ (ϕ is the Euler phi function) by listing all the generating intervals in the 14-chromatic scale, represented as elements of $\mathbb{Z}_{14}$. Indicate which pairs of generating intervals are inverse to each other and for each pair draw the circle of intervals which is based on one element of the pair in the clockwise direction, the other element of the pair in the counterclockwise direction.

Generating intervals are $\underbrace{[1], [13]}_{\text{inverses}}$, $\underbrace{[3], [11]}_{\text{inverses}}$, $\underbrace{[5], [9]}_{\text{inverses}}$

So $\phi(14) = 6$



8. Determine whether or not each of the following pairs forms a monoid. If so, is it also also a group? Justify your answers.

   (a) $(\{-1, 0, 1\}, \cdot)$ Closed with respect to $\cdot$ and associative, with identity element 1, so a monoid. 0 has no inverse, so not a group.

   (b) $(\mathbb{Z}_m, +)$ Associative with identity element $[0]$, hence a monoid. For $[l] \in \mathbb{Z}_m$, $[-l]$ is its inverse ($[l] + [-l] = [l-l] = [0]$) hence a group.

9. Explain why the functions $f(x) = 2^{x/12}$ and $g(x) = 12\log_2(x)$ are group homomorphisms, and why they are inverse to each other, thereby giving isomorphisms between the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$. Explain the connection of this with musical intervals.

$f(x+y) = 2^{\frac{x+y}{12}} = 2^{\frac{x}{12}+\frac{y}{12}} = 2^{\frac{x}{12}} \cdot 2^{\frac{y}{12}} = f(x)\cdot f(y)$

so f is a group homomorphism.

$g(x\cdot y) = 12\log_2(x\cdot y) = 12[\log_2 x + \log_2 y] = 12\log_2 x + 12\log_2 y$

$= g(x) + g(y)$, so g is a group homomorphism.

$f(g(x)) = 2^{\frac{g(x)}{12}} = 2^{\frac{12\log_2 x}{12}} = 2^{\log_2 x} = x$

$g(f(x)) = 12\log_2 f(x) = 12\log_2(2^{x/12}) = 12\cdot\frac{x}{12}\log_2 2$

$= x\log_2 2 = x\cdot 1 = x$.

So f and are inverse to each other, hence group isomorphisms.

f converts semitones to ratio, g does the opposite.

10. Suppose a 12-tone row chart begins: G, A$^\sharp$, D$^\flat$, F, B, ... Write the upper left 5×5 matrix of the resulting row chart. Then rewrite it replacing each note class with the element of $\mathbb{Z}_{12}$ which measures its modular interval from G.

G	A$^\sharp$	D$^\flat$	F	B
E	G	A$^\sharp$	D	G$^\sharp$
D$^\flat$	E	G	B	F
A	C	E$^\flat$	G	D$^\flat$
E$^\flat$	F$^\sharp$	A	D$^\flat$	G

[0]	[3]	[6]	[10]	[4]
[9]	[0]	[3]	[7]	[1]
[6]	[9]	[0]	[4]	[10]
[2]	[5]	[8]	[0]	[6]
[8]	[11]	[2]	[6]	[0]