EXAM III

Math 109 / Music 109A, Spring 2018

Name Solutions Id ____________

Each problem is worth 10 points.

1. Define a (commutative) ring R and show that $0 \cdot x = 0$, for any $x \in R$. Use this to show that $(-1) \cdot x = -x$, for any $x \in R$.

A ring R is a set with two laws of composition, $+$ and $\cdot$, such that $(R, +)$ is a commutative group and $(R, \cdot)$ is a commutative monoid. In addition, for all $a, b, c \in R$ we have $a \cdot (b + c) = a \cdot b + a \cdot c$

$0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$. Subtracting $0 \cdot x$, we get $0 = 0 \cdot x$

. To see that $(-1) \cdot x$ is the additive inverse of x, we write $(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = (-1+1) \cdot x = 0 \cdot x = 0$. So $(-1) \cdot x = -x$.

2. Determine whether these subsets of $\mathbb{Z}$ are ideals. If so, express them in the form $n\mathbb{Z}$, where n is a positive integer. Justify your answers.

(a) the positive integers

No. Does not contain 0, nor negatives.

(b) $100\mathbb{Z} + 30\mathbb{Z}$ $\gcd(100, 30) = 10$

so $10\mathbb{Z}$

3. Prove that there are infinitely many prime numbers.

Assume not. Then there are only finitely many primes $p_1, \cdots, p_r$. Let $N = p_1 \cdots p_r + 1$. Then N has a prime factor p_i. So $p_i \mid N$, i.e. $p_i \mid (p_1 \cdots p_r + 1)$. But $p_i \mid p_1 \cdots p_r$ so we must have $p_i \mid 1$, a contradiction.

4. List the units of the ring $\mathbb{Z}_{12}$, i.e., the elements of $\mathbb{Z}_{12}^*$, and explain the implication this has for keyboard intervals.

$\mathbb{Z}_{12}^{\times} = \{[1], [5], [7], [11]\}$. The corresponding keyboard intervals are the semitone, fourth, fifth, and major seventh. These are the generating intervals, i.e., those which, when iterated, give all note classes.

5. Give the prime factorizations of these integers, writing the primes in ascending order, as in $2^3 \cdot 3^1 \cdot 7^2$.

(a) 75 $= 3^1 \cdot 5^2$

(b) 64 $= 2^6$

(c) 242 $= 2 \cdot 11^2$

(d) 52 $= 2^2 \cdot 13^1$

(e) 14×10^{23} $= 2^{24} \cdot 5^{23} \cdot 7^1$

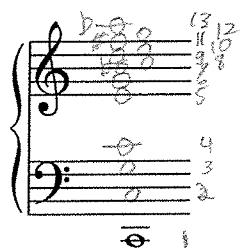
6. Prove that if $y = f(t)$ has period P, then $y = cf(t)$ has period P, but $y = f(ct)$ $(c \neq 0)$ has period P/c.

We have $f(t+P) = f(t)$

Then $cf(t+P) = cf(t)$ so $y = cf(t)$ has period P.

Also $f(c(t+\frac{P}{c})) = f(ct+P) = f(ct)$, so $y = f(ct)$ has period $\frac{P}{c}$.

7. On the staff system below, write the keyboard's best approximation for harmonics 1 through 13 for the indicated note. For the 7th, 11th, and 13th harmonics, indicate how sharp or flat (to the nearest cent) the keyboard's approximation is.



$12 \log_2 7 \approx 33.69$

so keyboard is ≈ 31 cents sharp

$12 \log_2 11 \approx 41.51$

so keyboard is ≈ 49 cents sharp

$12 \log_2 13 \approx 44.41$

so keyboard is ≈ 41 cents flat

8. Find the value γ for which the pitch associated to the periodic function $h(t) = d\sin(\gamma t + \beta)$, where t is time in seconds, is $E^\flat_4$.

$\gamma = 2\pi F$ $\quad F = 440 \cdot 2^{-6/12}$

$= 2\pi \cdot 440 \cdot 2^{-6/12} \approx 1954.87$

9. Find the period, frequency, amplitude, and phase shift for the function

$$g(t) = 3\sin(880\pi t) + 4\cos(880\pi t)$$

and express it in the form $d\sin(\alpha t+\beta)$, giving a decimal approximation for β.

$d = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = \boxed{5}$ amplitude

$2\pi F = 880\pi$, so $F = \boxed{440}$ (A_4) frequency

$g(t) = 5\left[\frac{3}{5}\sin(880\pi t) + \frac{4}{5}\cos(880\pi t)\right]$ $\boxed{P = \frac{1}{440} \approx 0.0023}$

$\cos\beta = \frac{3}{5}$, $\sin\beta = \frac{4}{5}$

$\boxed{g(t) = 5\sin(880\pi t + \beta)}$ $\beta = \boxed{\arcsin\frac{4}{5} \approx 0.93}$ phase shift

10. A certain vowel sound has a formant which amplifies frequencies within 350 Hz of 2900 Hz. An alto singer sings the vowel at A_3. Which harmonics are amplified?

$A_3 = 220$ Hz

$2550 \leq 220k \leq 3250$

$\frac{2550}{220} \leq k \leq \frac{3250}{220}$

$11.59 \leq k \leq 14.77$

$\boxed{k = 12, 13, 14}$