Homework 6
Math 109 / Music 109A, Spring 2010

Due Monday, April 19.

1. Find the value $\alpha$ for which the pitch associated to the periodic function $y = \sin(\alpha t)$, where $t$ is time in seconds, is:
   
   (a) middle C  
   (b) $A_4^2$  
   (c) $D_4^6$

2. Find the period, frequency, amplitude, and phase shift for these functions, and express each in the form $A \sin(\alpha t) + B \cos(\alpha t)$:
   
   (a) $f(t) = 5 \sin(30\pi t + \frac{\pi}{4})$

   (b) $g(t) = \sqrt{2} \sin(800t + \pi)$

3. Find the period, frequency, amplitude, and phase shift for these functions, and express each in the form $d \sin(\alpha t + \beta)$:
   
   (a) $f(t) = 4 \sin(300t) + 5 \cos(300t)$

   (b) $h(t) = -\sin(1500\pi t) + 3 \cos(1500\pi t)$
4. Suppose musical tone with pitch $B_4$ has harmonics 1, 3, 5 only, with amplitudes $1, \frac{1}{9}, \frac{1}{25}$, respectively, and phase shifts $0, \pi, -\frac{\pi}{2}$, respectively. Suppose also that the vertical shift $C$ is 0. Write its Fourier series in the form $\sum [A_k \sin(kt) + B_k \cos(kt)]$.

5. A certain soprano’s $ee$ vowel has a formant centered at 2900 Hz. What pitch should she sing in order for the fifth harmonic to be maximally amplified by this formant?

6. Two instruments play the pitches $A_2$ and $E_3$, making the interval of a keyboard fifth. Suppose they are playing the same kind of instrument, and that the instrument has a formant centered at 3000 Hz. Suppose the formant amplifies pitches within 400 Hz of its center. Identify the harmonics produced by each instrument which will be amplified by the formant, and give their frequencies. How many pairs of these frequencies are almost aligned? Could this “near alignment” be perfected by slightly adjusting the interval?
7. Let \( q(x) \) be defined by \( q(x) = \frac{1}{\pi} x - 1 \) on the interval \([0, 2\pi)\), extended to a periodic function on \( \mathbb{R} \) by periodicity. This a sawtooth wave. Its graph on \([0, 2\pi)\) is:

\[
\begin{array}{c}
\pi \\
2\pi
\end{array}
\]

\[ y = q(x) \]

The sound of this waveform is a harsh buzz. Its Fourier series is

\[
q(x) = -\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(kt).
\]

For \( k = 1 \) verify that this formula gives the correct sine coefficient. Hint: Mimic the computation for the square wave. You will need the formula

\[
\int_{0}^{2\pi} t \sin(kt) dt = -\frac{2\pi}{k}
\]

(which calculus students can verify using integration by parts).
8. Give the prime factorizations of these rational numbers as
\[
\frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}{q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}}
\]
with
\[
\{p_1, \ldots, p_r\} \cap \{q_1, \ldots, q_s\} = \emptyset,
\]
writing the primes of the numerator and denominator in ascending order, as in
\[
\frac{2^3 \cdot 7^2}{3^3 \cdot 13^3}.
\]
(a) \(\frac{150}{65}\) \hspace{1cm} (b) \(\frac{1000}{287}\) \hspace{1cm} (c) \(\frac{750}{980}\) \hspace{1cm} (d) \(\frac{512}{162}\) \hspace{1cm} (e) \(\frac{69}{289}\)

9. Give a direct proof that \(\sqrt{2}\) is irrational. Interpret this as a statement about a musical interval.

10. Show by multiplication and division in \(\mathbb{Q}^+\) that:
(a) a just major third plus a just minor third is a just fifth
(b) three just major thirds is not an octave
(c) a greater just whole tone plus a lesser just whole tone is a just major third