Affine Synthesis Onto L^p for $0 < p < 1$

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Motivation: Strang–Fix for $1 \leq p < \infty$

$\phi =$ analyzer, $\psi =$synthesizer, $f =$signal

$$T = \text{Analysis} : f \in L^p \mapsto \{\langle f, \phi(\cdot - k)\rangle\}_{k\in\mathbb{Z}} \in \ell^p$$
$$S = \text{Synthesis} : s \in \ell^p \mapsto \sum_k s_k \psi(\cdot - k) \in L^p$$

Similarly define S_j, T_j at scale 2^{-j}. *Limiting Reconstruction:*

if $\int_{\mathbb{R}} \phi \, dx = 1$ and $\sum_k \psi(x-k) \equiv 1$ then

$$\boxed{S_j T_j f \to f} \quad \text{in } L^p \text{ as } j \to \infty$$

Goal: prove same for $\boxed{0 < p < 1}$

Definitions

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}, \quad \text{metric } \|f-g\|_p^p$$
$$\|s\|_{\ell^p} = \left(\sum_{k\in\mathbb{Z}} |s_k|^p\right)^{1/p}, \qquad \text{metric } \|s-t\|_{\ell^p}^p$$

Synthesis $S : \ell^p \to L^p$

$$s \in \ell^p,\ \psi \in L^p \ \Rightarrow\ (Ss)(x) = \sum_k s_k \psi(x-k) \in L^p$$

Analysis $T : L^p \to \ell^p$

Assume analyzer ϕ is bounded with compact support, $\int_{\mathbb{R}} \phi \, dx = 1$.

Problem:
integral $\langle f, \phi(\cdot - k) \rangle$ need not exist, if $f \notin L^1$.

Solution:
write $\Theta(re^{i\theta}) = r^p e^{i\theta}$, so $|\Theta(z)| = |z|^p$ and

$$f \in L^p \quad \text{iff} \quad \Theta(f) \in L^1.$$

Define

$$(Tf)_k = \Theta^{-1}(\langle \Theta(f), \phi(\cdot - k) \rangle)$$

i.e., analyze a nonlinear stretch of the signal.

Then $T : L^p \to \ell^p$ is nonlinear and continuous.

Reconstruction when $\sum_k \psi(x-k) \equiv 1$

Synthesis and analysis at scale 2^{-j}:

$$(S_j s)(x) = \sum_k s_k \, 2^{j/p} \psi(2^j x - k)$$
$$(T_j f)_k = \Theta^{-1}(\langle \Theta(f), \phi(2^j \cdot -k) \rangle)$$

Limiting Reconstruction:

$$\boxed{S_j T_j f \to f} \quad \text{in } L^p \text{ as } j \to \infty$$

Reconstruction when $\sum_k \psi(x-k) \not\equiv 1$

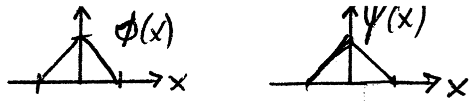
$$\|S_j T_j f - f\|_p \to \|\sum_k \psi(\cdot - k) - 1\|_{L^p[0,1]} \|f\|_p$$

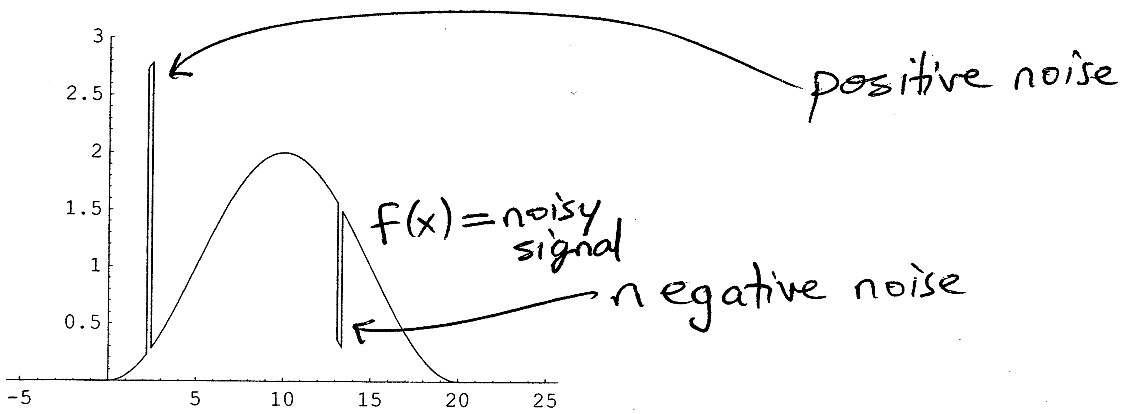
So if $\sum_k \psi(\cdot - k)$ is “close” to 1, then $S_j T_j f$ gets “close” to f, and by iterating (that is, using Open Mapping Theorem) we reconstruct $f = \sum_{j,k} c_{j,k} 2^{j/p} \psi(2^j x - k)$ for some sequence $\{c_{j,k}\} \in \ell^p(\mathbb{Z}_+ \times \mathbb{Z})$.

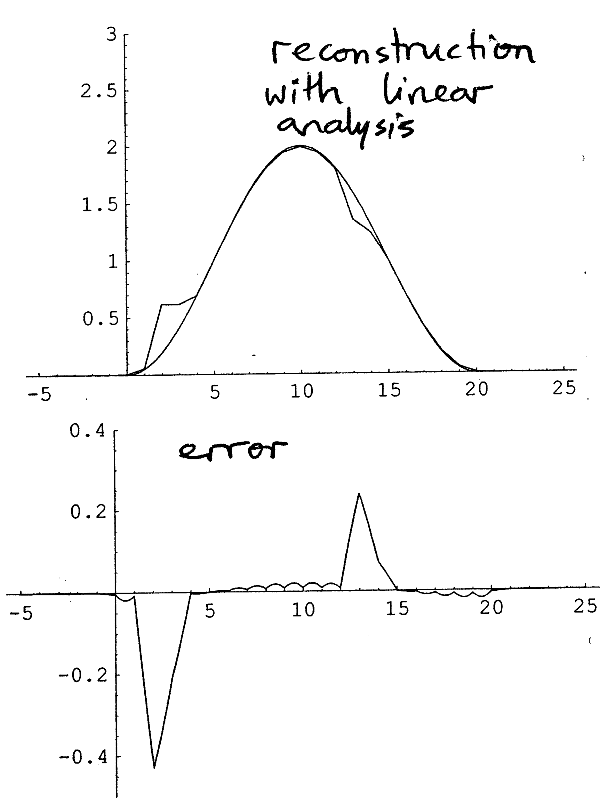
[cf. Filippov and Oswald, no control over $c_{j,k}$]

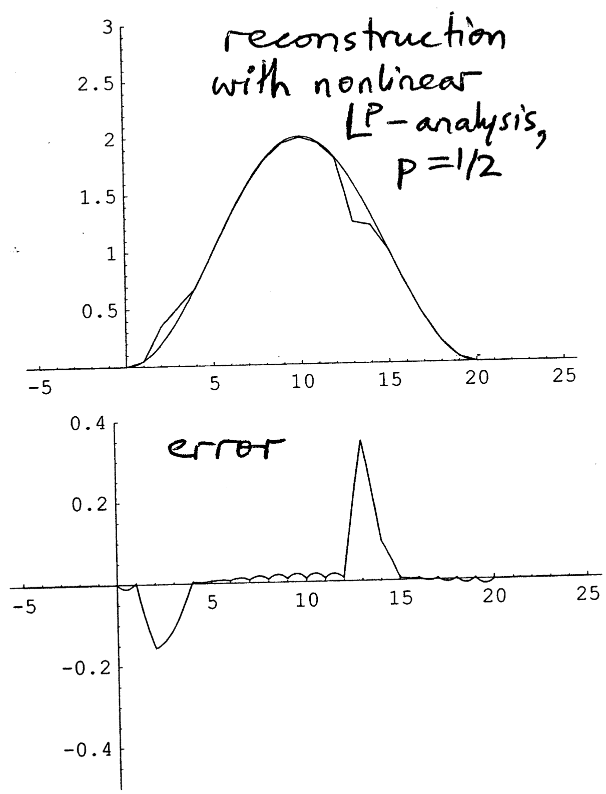
Conclusion Reconstruction using only small scales is possible even for some synthesizers with $\int_{\mathbb{R}} \psi \, dx = 0$, in L^p for $0 < p < 1$. *e.g.* $\psi =$Haar function.

Example









Conclusion

Compared to linear analysis $\langle f, \phi(\cdot - k)\rangle$, the nonlinear L^p-analysis $\Theta^{-1}(\langle \Theta f, \phi(\cdot - k)\rangle)$ performs better for positive signals with positive noise, but worse for positive signals with negative noise.

Applications?

- Are there discrete settings where L^p-analysis could be useful?

  (ie $f = \{f(n)\}_{n\in\mathbb{Z}}$, $(Tf)(k) = \Theta^{-1}\left(\sum_n (\Theta f)(n)\,\phi(n-k)\right)$,
  reconstruction $(STf)(n) = \sum_k (Tf)(k)\,\psi(n-k)$)

- Are there practical applications where it is better to approximate the signal/image in L^p for some $p<1$, rather than for $p=2$?

  Do the above methods apply?