Lei Gao   Math 417
Hwk #1

2. If f is a bijection, we define $g: Y \to X$ as follows: for $\forall y \in Y$, since f is onto, $\exists\, x \in X$ s.t. $f(x) = y$, let $g(y) := x$.
g is well defined, if $g(y_1) = x_1$, $g(y_2) = x_2$, $\forall\, y_1, y_2 \in Y$ and $x_1, x_2 \in X$ s.t. $f(x_1) = y_1$, $f(x_2) = y_2$, when $y_1 = y_2$, since f is one to one, $x_1 = x_2$. This proved g is a function from Y to X. By definition, $g \circ f = i_X$, $f \circ g = i_Y$, f has an inverse.
If f has an inverse, then $\exists\, g: Y \to X$ s.t. $g \circ f = i_X$, $f \circ g = i_Y$.
For $\forall\, y \in Y$, $f(g(y)) = y$, f is onto. If $f(x_1) = y_1$, $f(x_2) = y_2$, and $y_1 = y_2$, then $x_1 = g \circ f(x_1) = g(y_1) = g(y_2) = g \circ f(x_2) = x_2$, so f is one to one. So f is a bijection.

3(a) $x \in \underline{\lim} A_n \Leftrightarrow x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \Leftrightarrow \exists\, n_0,\ 1 \le n_0 < \infty,\ x \in \bigcap_{k=n_0}^{\infty} A_k$
$\Leftrightarrow \exists\, n_0, x \in A_k$ for $\forall\, k \geq n_0 \Leftrightarrow x$ is in all but finitely many of A_n.

(b) $x \in \overline{\lim} A_n \Leftrightarrow x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \Leftrightarrow$ for $\forall\, n$, $1 \le n < \infty$, $x \in \bigcup_{k=n}^{\infty} A_k$
$\Leftrightarrow$ for a given n, $\exists\, k(n)$, $n \le k(n) < \infty$, $x \in A_{k(n)}$
$\Leftrightarrow x$ is in infinitely many of the A_n.

(c)(i) $\bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{k=n}^{\infty} A_k$ (for $\forall n$) $\subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \underline{\lim} A_n$
(ii) For $\forall\, x$ in $\underline{\lim} A_n$, x is in all but finitely many of the A_n, then x is in infinitely many of the A_n, $x \in \overline{\lim} A_n$.
This proves $\underline{\lim} A_n \subseteq \overline{\lim} A_n$
(iii) $\overline{\lim} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = (\bigcup_{k=1}^{\infty} A_k) \cap (\bigcap_{n=2}^{\infty} \bigcup_{k=n}^{\infty} A_k) \subseteq \bigcup_{n=1}^{\infty} A_n$

(d) De Morgan's laws are true for countably many sets.
$\underline{\lim} (X \setminus A_n) = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} (X \setminus A_k) = \bigcup_{n=1}^{\infty} (X \setminus \bigcup_{k=n}^{\infty} A_n) = X \setminus (\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k)$
$= X \setminus \overline{\lim} A_n$

(e) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, $\underline{\lim} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n$, $\overline{\lim} A$

$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k$, so $\underline{\lim} A_n = \overline{\lim} A_n$

If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, $\underline{\lim} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcap_{k=1}^{\infty} A_k$, $\overline{\lim} A_n = \bigcap_{n=1}^{\infty}$

$= \bigcap_{n=1}^{\infty} A_n$, so $\underline{\lim} A_n = \overline{\lim} A_n$.

4. f is onto since for $\forall\, x \in X$, $\exists\, n \in \mathbb{N}_0$ s.t. $f^n(x) = x$, t

$f(f^{(n-1)}(x)) = x$. If $f(x_1) = y_1$, $f(x_2) = y_2$, and $y_1 = y_2$

$\exists$, $n_1, n_2 \in \mathbb{N}_0$ for ~~y_1 (or y_2)~~, x_1, x_2 respectfully s.t

~~$f^{n_0}(y_1) = f^{n_0}(y_2)$~~, $f^{n_1}(x_1) = x_1$, $f^{n_2}(x_2) = x_2$, But $y_1 = y_2$

So $f^n(x_1) = f^n(x_2)$ for $\forall\, n \geq 1$, We have $x_1 = f^{n_1 n_2}(x_1) = f^{n_1 n}$

$= x_2$, f is one to one. f is a bijection.