417 Homework Set # 2. Due September 13, 2001.

1. Prove that there is no continuous bijection from $[0,1]$ onto $(0,1)$.

2. The Hausdorff maximality principle says:

Let $(X, \preceq)$ be a partially ordered set. Then there exists a maximal chain in X.

The proof that the axiom of Choice is equivalent to both Zermelo's Well-ordering principle (every set can be given a partial order that is a well-ordering) and to the Hausdorff maximality principle is sketched in Munkres. It is quite long.

Prove that Zorn's lemma is equivalent to the Hausdorff maximality principle.

3. Let Ω be the first uncountable ordinal. Let X be a countable subset of Ω. Prove that X has an upper bound in Ω.

4. For each statement, prove or give a counter-example.

a) If X is a collection of pairwise disjoint circles in the plane, then X is countable.

b) If X is a collection of pairwise disjoint disks in the plane, then X is countable.

5. Writing $A \sim B$ to mean A and B have the same cardinality, prove or give a counter-example to each of the following statements.

a) If $A \sim B$ and $C \sim D$, then $A \cup B \sim C \cup D$.

b) If A and C are infinite, and $A \sim B$ and $C \sim D$, then $A \cap B \sim C \cap D$.

c) If $A \setminus B \sim B \setminus A$, then $A \sim B$.

d) If $A,\ B$ and C are non-empty, and $A \times B \sim A \times C$, then $B \sim C$.

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1. Proof: Suppose f is a continuous bijection from $[0,1]$ to $(0,1)$. We may assume $f(0) < f(1)$ without lost of generality. $f(0) \in (0,1)$, then $\exists\ b$ s.t. $0 < b < f(0)$, f is onto so $\exists\ a \in (0,1)$ s.t. $f(a) = b$. Consider the closed interval $[a,1]$, since $f(a) < f(0) < f(1)$, by Intermediate Value Theorem, $\exists\ c \in (a,1)$ s.t. $f(c) = f(0)$. But $c \neq 0$ contradicting to f one to one.

4 (a) Let X be the collection of circles centered at origin with irrational radii. Since irrational numbers are uncountable, X is uncountable.

(b) Proof: Since $\mathbb{Q} \times \mathbb{Q}$ is dense on $\mathbb{R}^2$, each disk contains at least one point with rational coordinates. For every element in X, pick one such point. Then we have an injective mapping from X to $\mathbb{Q} \times \mathbb{Q}$. So $\mathrm{card}(X) \leq \mathrm{card}(\mathbb{Q} \times \mathbb{Q})$. Since $\mathbb{Q} \times \mathbb{Q}$ is countable. X is countable.

5 (a) Let $A = B = \mathbb{N}$, $C = D = \mathbb{R}$, then $A \sim B$, $C \sim D$. But $A \cup B = \mathbb{N}$, $C \cup D = \mathbb{R}$, $\mathbb{N} \nsim \mathbb{R}$.

(b) Let $A = 2\mathbb{N}$, $B = \mathbb{N} - 2\mathbb{N}$, $C = D = \mathbb{R}$, then $A \sim B$, $C \sim D$. But $A \cap B = \phi$, $C \cap D = \mathbb{R}$, $\mathrm{card}(\phi) \neq \mathrm{card}(\mathbb{R})$.

(c) Proof: Since $A \setminus B \sim B \setminus A$, there exists a bijection f from $A \setminus B$ to $B \setminus A$. Define F from A to B by $F|_{A \setminus B} = f$, $F|_{A \cap B} = id_{A \cap B}$. F has inverse as G defined by $G|_{B \setminus A} = f^{-1}$, $G|_{A \cap B} = id_{A \cap B}$. F is bijection from A to B. $A \sim B$.

(d) Let $A = B = \mathbb{N}$, $C = \{1\}$, then $A \times B \sim A \times C$, But $\mathrm{card}(B) \neq \mathrm{card}(C)$.

3. Page 66. Thm 10.3

2. Page 71 & Exercise 5-7.