417 Homework Set # 3. Due September 27, 2001.

1. Suppose that $f : [0,1] \to \mathbb{R}$ has the following property:
There is a constant M such that, for every finite set $\{x_1, x_2, \ldots, x_n\} \subseteq [0,1]$,

$$|\sum_{i=1}^{n} f(x_i)| \; < \; M.$$

a) Give an example of such a function f where range(f) is infinite.
b) Prove that for such a function, range(f) is countable.

2. Let A, B, C, D be sets. Using the symbol $\sim$ to denote "has the same cardinality as", prove:

$$A \sim C \text{ and } B \sim D \;\Rightarrow\; A \times B \sim C \times D.$$

3. Find the cardinality of each of the following sets.
a) The set of all convergent sequences of real numbers.
b) The set of all straight lines in the plane that contain at least two points with both coordinates rational.
c) The set of all sequences $f : \mathbb{N} \to \mathbb{N}$ that are eventually constant.
d) The set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$.
e) The set of all strictly increasing sequences $f : \mathbb{N} \to \mathbb{N}$.

4. On $C[0,1]$ define d_1 and d_∞ by

$$\begin{aligned} d_1(f,g) &= \int_0^1 |f(x) - g(x)|dx \\ d_\infty(f,g) &= \sup_{0 \le x \le 1} |f(x) - g(x)|. \end{aligned}$$

Prove both d_1 and d_∞ are metrics.

5. Let (X,d) be a metric space. Prove that

$$d'(x,y) := \frac{d(x,y)}{1 + d(x,y)}$$

defines a metric on X with values always bounded by 1.

6. (Bonus problem if you know a little about complex numbers). On the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, define the pseudo-hyperbolic metric by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w} z} \right| .$$

Prove that ρ is a metric on the unit disk.

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1 (a) Define f as follows: $f(x) = 2^{-n}$ if $x = \frac{1}{n}$, $n \in \mathbb{N}$, $f(x) = 0$ otherwise. Since $\sup |\sum_{i=1}^{n} f(x_i)| = \lim_{n\to\infty}(\sum_{i=1}^{n} 2^{-i}) = 1$, then $M = 2$ is a required constant s.t. for every finite $\{x_1, x_2, \cdots x_n\} \subseteq [0,1]$, $|\sum_{i=1}^{n} f(x_i)| < M$.

(b) Proof: Suppose range(f) is uncountable. We may assume it has uncountably many positive numbers. Divide $(0, +\infty)$ into intervals $(1, +\infty)$ and $(\frac{1}{n+1}, \frac{1}{n}]$, $n = 1, 2, \cdots$. Then at least one of them must contains uncountably many values of range. So we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [0,1]$, and $f(x_n) > \delta$, $\delta > 0$ fixed. But we will have $\sup|\sum_{i=1}^{n} f(x_i)| = +\infty$, impossible.

2 Proof: If $A \sim C$, then $\exists$ a bijection $f: A \to C$. And $B \sim D$, so $\exists$ a bijection $g: B \to D$. Define a function h from $A \times B$ to $C \times D$ by $h: (a,b) \longmapsto (f(a), g(b))$ where $a \in A$, $b \in B$.
h is one to one: if $h(a,b) = h(a',b')$, then $(f(a), g(b)) = (f(a'), g(b'))$. So we have $f(a) = f(a')$, $g(b) = g(b')$. Since f, g are bijection, $a = a'$, $b = b'$. Then $(a,b) = (a',b')$.
h is onto: for $\forall (c,d) \in C \times D$, f, g are onto, so $\exists$ a, b with $a \in A$, $b \in B$ s.t. $f(a) = c$, $g(b) = d$, then $h(a,b) = (c,d)$. h is a bijection from $A \times B$ to $C \times D$, so $A \times B \sim C \times D$.

4 (a) Proof: For $\forall f, g, h \in C[0,1]$, we have

(i) $d_1(f,f) = \int_0^1 |f(x) - f(x)|\,dx = 0$.

(ii) $d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx = \int_0^1 |g(x) - f(x)|\,dx = d_1(g,f)$.

(iii) If $f \neq g$, then $\exists\, x \in (0,1)$ s.t. $f(x) \neq g(x)$ or $|f(x) - g(x)| > \delta$ with $\delta > 0$. Since f, g are continuous, $\exists\, \varepsilon > 0$ s.t. $|f(x) - g(x)| > \delta$ on interval $(x-\varepsilon, x+\varepsilon)$ with $(x-\varepsilon, x+\varepsilon) \subset (0,1)$. So $d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx > \int_{x-\varepsilon}^{x+\varepsilon} \delta\,dx = 2\varepsilon\delta > 0$.

(iv) $d_1(f,h) = \int_0^1 |f(x) - h(x)|\,dx = \int_0^1 |f(x) - g(x) + g(x) - h(x)|\,dx \leq$

$\leq \int_0^1 |f(x) - g(x)| + |g(x) - h(x)| \, dx = d_1(f, g) + d_1(g, h)$.

So d_1 is a metric.

(b) (i) $d_\infty(f, f) = \sup_{0 \leq x \leq 1} |f(x) - f(x)| = 0$.

(ii) $d_\infty(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)| = \sup_{0 \leq x \leq 1} |g(x) - f(x)| = d_\infty(g, f)$.

(iii) If $f \neq g$, then $\exists\, x \in [0,1]$ s.t. $f(x) \neq g(x)$. So $d_\infty(f, g) =$
$= \sup_{0 \leq x \leq 1} |f(x) - g(x)| > 0$.

(iv) $d_\infty(f, h) = \sup_{0 \leq x \leq 1} |f(x) - h(x)| = \sup_{0 \leq x \leq 1} |f(x) - g(x) + g(x) - h(x)| \leq$
$\leq \sup_{0 \leq x \leq 1} (|f(x) - g(x)| + |g(x) - h(x)|) \leq \sup_{0 \leq x \leq 1} |f(x) - g(x)| + \sup_{0 \leq x \leq 1} |g(x) - h(x)|$
$= d_\infty(f, g) + d_\infty(g, h)$.

So d_∞ is a metric.

5 Proof: Let $x, y, z \in X$ arbitrary.

(i) $d'(x, x) = d(x,x)/(1 + d(x,x)) = 0$.

(ii) $d'(x, y) = d(x,y)/(1 + d(x,y)) = d(y,x)/(1 + d(y,x)) =$
$= d'(y, x)$.

(iii) If $x \neq y$, $d(x,y) > 0$, then $d'(x,y) = d(x,y)/(1 + d(x,y)) >$

(iv) Since $\left(\frac{x}{1+x}\right)' = \frac{1}{(1+x)^2}$, $\frac{x}{1+x}$ is strictly increasing on $[0, \infty)$. Because $d(x, z) \leq d(x, y) + d(y, z)$,

$$\frac{d(x,z)}{1 + d(x,z)} \leq \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)} = \frac{d(x,y)}{1 + d(x,y) + d(y,z)} + \frac{d(y,z)}{1 + d(x,y) + d(y,z)}$$

$$\leq \frac{d(x,y)}{1 + d(x,y)} + \frac{d(y,z)}{1 + d(y,z)}.$$ So $d'(x,z) \leq d'(x,y) + d'(y,z)$

Since $d(x,y)/(1 + d(x,y)) < 1$ whenever $d(x,y) > 0$.
d' is a metric on X with value bounded by 1.

3.(a) Write $\mathrm{card}(\mathbb{N}) = \aleph_0$, $\mathrm{card}(\mathbb{R}) = c$. Let S be the set of all convergent sequences of real numbers. For any $r \in \mathbb{R}$ there exists $\{x_n\}$ with $x_n = r$, $n = 1, 2, \cdots$. $\{x_n\} \in S$, so $\mathrm{card}(S) \geq c$. On the other hand, given an element in S, $\{x_n\}_{n=1}^\infty$, since $\mathbb{R}$ is homeomorphic to $(0,1)$, we can transform it into a sequence of real number which are in $(0,$

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For $\forall n$, $x_n = 0.a_{n1}a_{n2}\cdots a_{nk}\cdots$ with $a_{nk} \in \mathbb{N}$, $0 \le a_{nk} \le 9$
In case x_n is rational, then from some k, $a_{nk} = 0$ or
we use the former one. We construct a real number
from x_n's as follows: $0.a_{11}a_{12}a_{21}\cdots a_{1k}\cdots a_{(k+1-i)i}\cdots a_{k1}\cdots$

$$\begin{array}{l} 0.\ a_{11} \to a_{12}\ a_{21}\ \cdots\ a_{1k}\ \cdots \\ 0.\ a_{21}\ a_{22}\ \cdots \\ \vdots \\ 0.\ a_{k1}\ \cdots \end{array}$$

From the process of construction, this mapping is injective
$card(S) \le c$. So $card(S) = c$.

(b) Let S be the set in the problem. Given a line l in S
$\exists$ two points with both coordinates rational that lie on
So we have a surjective mapping from $(\mathbb{Q}\times\mathbb{Q})\times(\mathbb{Q}\times\mathbb{Q})$
to S. $card(S) \le \aleph_0$. On the other hand, given
$\forall (a,b) \in \mathbb{Q}\times\mathbb{Q}$, we have a line l that passes poi
$(a,0)$ with tangent b. Clearly $l \in S$, then we ge
an injective mapping from $\mathbb{Q}\times\mathbb{Q}$ to S. $card(S) \ge \aleph_0$
So $card(S) = \aleph_0$.

(c) Let S be the set in the problem. The elements in S
are 1-1 correspondent with elements in $\mathbb{N}^k$ with
$k = 1, 2, \cdots$ $card(\mathbb{N}^k) = \aleph_0$. $card(S) = card(\bigcup_{k=1}^{\infty} \mathbb{N}^k)$
$= \aleph_0$.

(d) Let S be the set of all differentiable functions $f: \mathbb{R} \to \mathbb{R}$.
For $\forall r \in \mathbb{R}$, $f \equiv r$ a constant function is differentiable
So $card(S) \ge c$. On the other hand, for each
$f \in S$, we can fix a subset of $\mathbb{Q}\times\mathbb{Q}$ given by:
$\{(x,y) \in \mathbb{Q}\times\mathbb{Q},\ x$ is in the domain of f. $y \le f(x)\}$.
f is differentiable, then continuous, so this mapping
is injective from S to $\mathcal{P}(\mathbb{Q}^2)$. $card(S) \le c$. So $card(S) =$

(e) Let S be the set of all strictly increasing sequence $f: \mathbb{N} \to \mathbb{N}$. For each $\{n_k\}_{k=1}^{\infty}$ in S, $n_1 < n_2 < \cdots < n_k < \cdots$, we have a real number r in $(0,1]$ that corresponds to it: $r = \sum_{k=1}^{\infty} \frac{1}{2^{n_k}}$. $\text{card}(S) \leq \text{card}((0,1]) = c$. On the other hand, for each r in $(0,1]$, $r = \sum_{k=1}^{\infty} \frac{1}{2^{n_k}}$. If r is rational, $r = \sum_{k=1}^{m} \frac{1}{2^{n_k}} + \frac{1}{2^{n_m+1}}$ or $r = \sum_{k=1}^{m} \frac{1}{2^{n_k}} + \sum_{i=1}^{\infty} \frac{1}{2^{n_m+1+i}}$, for some $m \in \mathbb{N}$. We always use the latter one. So both rational and irrational number can be written as a infinite sum. Then $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence. The mapping is injective. $\text{card}(S) \geq c$. So $\text{card}(S) = c$.