417 Homework Set # 4. Due October 4, 2001.

1. Show that for every p and q in the range $[1, \infty]$, the metrics d_p and d_q on $\mathbb{R}^n$ induce the same topologies.

2. For $1 \leq p < \infty$, define

$$\ell^p = \{(a_1, a_2, \ldots) : \sum_{i=1}^{\infty} |a_i|^p < \infty\}.$$

Show that if $p < q$, then $\ell^p \subsetneq \ell^q$.

3. The proof we gave in class that d_p is a metric on $\mathbb{R}^n$ extends immediately to show d_p is a metric on ℓ^p. Prove that on ℓ^1, the metrics d_1 and d_2 do *not* induce the same topologies.

4. Let (X_i, d_i) be metric spaces for $1 \leq i \leq n$. Prove that, for each p in the range $[1, \infty)$, the function

$$D_p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \left[\sum_{i=1}^{n} [d_i(x_i, y_i)]^p\right]^{\frac{1}{p}}$$

defines a metric on $X_1 \times \ldots \times X_n$. How should D_∞ be defined?

5. Let $X = \{\frac{1}{n} : n \in \mathbb{N}_0\}$ and let d be the usual metric on $\mathbb{R}$. Let s denote the discrete metric on X. Prove that d and s induce the same topology on X.

6. Prove that the only clopen sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.

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1. Proof: On a set X, we say two metrics d_1, d_2 are equivalent iff $\exists\, k, K$, with $0 < k < K < \infty$, s.t. $kd_1 \le d_2 \le K d_1$. Written as $d_1 \sim d_2$. Clearly $d_1 \sim d_1$. If $d_1 \sim d_2$, then $kd_1 \le d_2 \le Kd_1$ for k, K with $0<k<K<\infty$. So $\frac{1}{K} d_2 \le d_1 \le \frac{1}{k} d_2$, $d_2 \sim d_1$. If $d_1 \sim d_2$, $d_2 \sim d_3$, then $k_1 d_1 \le d_2 \le K_1 d_1$, $k_2 d_2 \le d_3 \le K_2 d_2$ for $0<k_1<K_1<\infty$, $0<k_2<K_2<\infty$. So $k_1 k_2 d_1 \le d_3 \le K_1 K_2 d_1$, $d_1 \sim d_3$. The relation is equivalent relation. For $\forall\, U \subset X$, metrics d_1, d_2 induce topologies $\mathcal{T}_1, \mathcal{T}_2$. Suppose $d_1 \sim d_2$, i.e. $kd_1 \le d_2 \le K d_1$ with $0<k<K<\infty$. If U is open in $\mathcal{T}_1$, for $\forall\, x \in U$, $\exists\, \delta > 0$ s.t. $B_{d_1}(x, \delta) \subset U$. For this x, $\forall\, y \in B_{d_2}(x, k\delta)$ we have $d_1(x,y) \le k^{-1} d_2(x,y) < \delta$, $B_{d_2}(x, k\delta) \subset U$. U is open in $\mathcal{T}_2$. Similarly, if U is open in $\mathcal{T}_2$ then U is open in $\mathcal{T}_1$. d_1, d_2 induce the same topology. Now $X = \mathbb{R}^n$, we know $d_\infty \le d_p \le d_1$ for $p \in [1, \infty]$. And $d_1 \le n d_\infty$, since $|x_i - y_i| \le \max_{1 \le i \le n} |x_i - y_i|$ if $x = (x_i)_1^n$, $y = (y_i)_1^n$, $d_1(x,y) = \sum_{i=1}^n |x_i - y_i|$, $d_\infty = \max_{1\le i \le n} |x_i - y_i|$. Then $d_\infty \sim d_p$, and $d_\infty \sim d_q$ for $p, q \in [1, \infty]$. So $d_p \sim d_q$, they induce the same topologies.

2. Proof: Suppose $p < q$. For $\forall\, (a_i)_1^\infty \in \ell^p$, $p \in [1, \infty)$. Let $\sum_{i=1}^\infty |a_i|^p = A < \infty$, then $\lim_{i\to\infty} |a_i|^p = \lim_{i\to\infty} \left(\sum_{k=1}^{i} |a_k|^p - \sum_{k=1}^{i-1} |a_k|^p \right) = 0$. And $p \ge 1$, so $|a_i| < 1$ if i big enough. Suppose $|a_i| < 1$ if $i \ge m$. $\sum_{i=m}^\infty |a_i|^q = \sum_{i=m}^\infty |a_i|^p \cdot |a_i|^{q-p} < \sum_{i=m}^\infty |a_i|^p < \infty$, $p < q$. So $(a_i)_1^\infty \in \ell^q$. $\ell^p \subseteq \ell^q$. Next to show $\ell^p \subsetneq \ell^q$. Let $a_n = 1/n^{\frac{1}{p}}$. Then $\sum_{n=1}^\infty |a_n|^p = \sum_{n=1}^\infty \frac{1}{n}$, diverges; $\sum_{n=1}^\infty |a_n|^q = \sum_{n=1}^\infty \left(\frac{1}{n}\right)^{q/p} < \infty$. $(a_n)_1^\infty \in \ell^q$ but $(a_n)_1^\infty \notin \ell^p$. This proves $\ell^p \subsetneq \ell^q$.

3. Proof: ~~Let $0 = (0, 0, \dots)$, $x_n = (\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}, 0, \dots)$ with n times $\frac{1}{n}$. Then $\exists\, 0 \in \ell^1$, $\exists\, \varepsilon = \frac{2}{3} > 0$, we have $B_{d_2}(0, \frac{2}{3})$. $\{x_n\}_{n=2}^\infty$ is a convergent sequence in $B_{d_2}(0, \frac{2}{3})$, but $d_1(x_n, 0) = 1$ for $\forall\, n$, $n = 2, 3, \dots$. So if $\delta \in (0,1)$ arbitrary, $B_{d_1}(0, \delta) \not\supseteq B_{d_2}(0, \frac{2}{3})$~~

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~~This means d_1, d_2 do not induce the same topologies.~~

4(a) Proof: (i) (ii) (iii) Easy.

(iv) Let $x = (x_1, \cdots x_n)$, $y = (y_1, \cdots y_n)$, $z = (z_1, \cdots z_n)$. Then $d_i(x_i, z_i)^p \le [d_i(x_i, y_i) + d_i(y_i, z_i)] \cdot d_i(x_i, z_i)^{p-1}$. So $\sum_{i=1}^n d_i(x_i, z_i)^p$ $\le \sum_{i=1}^n d_i(x_i, y_i) \cdot d(x_i, z_i)^{p-1} + \sum_{i=1}^n d_i(y_i, z_i) \cdot d_i(x_i, z_i)^{p-1}$. Apply Hölder's inequality, let $q = p/(p-1)$. Then we have

$$\sum_{i=1}^n d_i(x_i, z_i)^p \le \left[\sum_{i=1}^n d_i(x_i, y_i)^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n d_i(x_i, z_i)^{(p-1)q}\right]^{\frac{1}{q}} + \left[\sum_{i=1}^n d_i(y_i, z_i)^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^n d_i(x_i, z_i)^{(p-1)q}\right]$$

$$\text{So } D_p(x, z)^p \le D_p(x, y)\, D_p(x, z)^{\frac{p}{q}} + D_p(y, z)\, D_p(x, z)^{\frac{p}{q}}$$

$$D_p(x, z)^{p - \frac{p}{q}} \le D_p(x, y) + D_p(y, z),$$

$$\therefore D_p(x, z) \le D_p(x, y) + D_p(y, z) \quad \text{since } \frac{1}{p} + \frac{1}{q} = 1.$$

(Solution given by Aaron, Naveen, Mike, Paul, Bryce)

(b) Define $D_\infty(x, y) = \max_{1 \le i \le n} d_i(x_i, y_i)$.

5. Proof: s induces discrete topology. Want to show d induces also discrete topology on X. For $\forall x_n = \frac{1}{n} \in X$, choose $\varepsilon > 0$ s.t. $\varepsilon = \frac{1}{2} \min(|x_{n-1} - x_n|, |x_n - x_{n+1}|)$. Then $B_d(x_n, \varepsilon)$ is open and contains only x_n. So d, s induce the same topology on X

6. Proof: Clearly ϕ, $\mathbb{R}$ are clopen sets in $\mathbb{R}$. Want to show these are the only clopen sets in $\mathbb{R}$. Suppose not, $\exists A \subsetneq \mathbb{R}$ $A \neq \phi$, A is clopen. Let $B = \mathbb{R} - A$, B is closed and nonempty since A is open and $A \neq \mathbb{R}$. Then $A \cap B = \phi$, $A \cup B = \mathbb{R}$. For $\forall a \in A$, $b \in B$, we may assume $a < b$. Let $\hat{A} = A \cap [a, b]$, $\hat{B} = B \cap [a, b]$. Then $\hat{A}, \hat{B}$ are nonempty closed sets in $\mathbb{R}$, with $\hat{A} \cap \hat{B} = \phi$, $\hat{A} \cup \hat{B} = [a, b]$. $\hat{A}$ has upper bound b, then least upper bound, let it be $\hat{b}$. $\hat{A}$ closed, so $\hat{b} \in \hat{A}$. If $\hat{b} = b$, then $b \in \hat{A} \cap \hat{B}$, a contradiction. So $\hat{b} < b$, then $(\hat{b}, b] \subset \hat{B}$. $\hat{B}$ is closed, so $\hat{b} \in \hat{B}$, but $\hat{b} \in \hat{A}$, contradicting to $\hat{A} \cap \hat{B} = \phi$.

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3. Proof: Want to show $\exists$ a set which is open in $\mathcal{T}_1$ but not open in $\mathcal{T}_2$, where $\mathcal{T}_1$, $\mathcal{T}_2$ are the topologies induced by d_1, d_2 respectively. Let $0 = (0, 0, \cdots)$, $x_n = (\frac{1}{n}, \cdots \frac{1}{n}, 0, \cdots)$ with n times $\frac{1}{n}$. $d_1(x_n, 0) = 1$, but $d_2(x_n, 0) \longrightarrow 0$, if $n \to \infty$. Let $U = B_{d_1}(0, \frac{1}{2})$, an open ball centered at 0 with radius $\frac{1}{2}$. $0 \in U$, for $\forall \varepsilon > 0$, $\exists x_n$, $x_n \in B_{d_2}(0, \varepsilon)$. But $x_n \notin B_{d_1}(0, \frac{1}{2})$ since $d_1(x_n, 0) = 1$. Then $B_{d_2}(0, \varepsilon) \nsubseteq B_{d_1}(0, \frac{1}{2})$ for $\forall \varepsilon > 0$. U is not open in $\mathcal{T}_2$.