417 Homework Set # 5. Due October 11, 2001.

1. Let (X, d) be a metric space. A function $f : X \to X$ is continuous at a if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. if } d(x,a) < \delta \text{ then } d(f(x), f(a)) < \varepsilon.$$

Describe those functions $f : X \to X$ with the following properties:

$$\exists \delta \; \forall \varepsilon > 0 \quad \text{s.t. if} \quad d(x,a) < \delta \text{ then } d(f(x), f(a)) < \varepsilon. \qquad (1)$$
$$\forall \varepsilon > 0 \; \forall \delta > 0 \quad \text{s.t. if} \quad d(x,a) < \delta \text{ then } d(f(x), f(a)) < \varepsilon. \qquad (2)$$
$$\exists \varepsilon > 0 \; \exists \delta > 0 \quad \text{s.t. if} \quad d(x,a) < \delta \text{ then } d(f(x), f(a)) < \varepsilon. \qquad (3)$$

2. Let A and B be subsets of a metric space (X, d). Which of the relations $=$, $\subseteq$, $\supseteq$ holds between $\overline{A \cap B}$ and $\bar{A} \cap \bar{B}$? Between $\overline{A \setminus B}$ and $\bar{A} \setminus \bar{B}$?

3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x+y) = f(x) + f(y) \; \forall \, x, y \in \mathbb{R}.$$

4. Prove that the composition of two continuous functions is continuous.

5. Give an example of two disjoint closed sets A and B in a metric space (X, d) with $dist(A, B) = 0$.

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1\. (1) Since ε is arbitrary, $d(f(x), f(a)) = 0$ for any x that satisfies $d(x, a) < \delta$, δ fixed. So $f(x)$ is constant on the open ball centered at a with radius δ. f is locally const.

(2) For any $x \in X$, let $\delta = 2d(x, a)$, then $d(x, a) < \delta$. So $d(f(x), f(a)) < \varepsilon$, $\varepsilon > 0$ arbitrary, for this fixed x. By the same reason in (1), $f(x) = f(a)$. $f \equiv$ const.

(3) f maps open ball centered at a with radius δ into the open ball centered at $f(a)$ with radius ε. f is locally bounded at ~~near~~ a

2 (1) Since $(A \cap B)' \subseteq A'$, $(A \cap B)' \subseteq B'$, $(A \cap B)' \subseteq A' \cap B'$. By Theorem 17.6. $\overline{A \cap B} = (A \cap B) \cup (A \cap B)' \subseteq (A \cap B) \cup (A' \cap B')$
$= (A \cup A') \cap (B \cup B') \cap (A \cup B') \cap (B \cup A') = (A \cup A') \cap (B \cup B') = \bar{A} \cap \bar{B}$.
The second last equality holds because $(A \cup A') \cap (B \cup B') \subseteq A \cup B'$ and similarly $B \cup A'$. So $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. The equality doesn't always hold. For example, $A = (0, 1)$, $B = (1, 2)$, $\overline{A \cap B} = \phi$, $\bar{A} \cap \bar{B} = \{1\}$. (In case $X = \mathbb{R}$, d usual metric)

(2) $\bar{A} \setminus \bar{B} = \bar{A} \cap \bar{B}^c = \bar{A} \cap \operatorname{int} B^c \subseteq \overline{A \cap \operatorname{int} B^c} = \overline{A \cap B^c} = \overline{A \setminus B}$.
So $\bar{A} \setminus \bar{B} \subseteq \overline{A \setminus B}$. The equality doesn't always hold.
In $\mathbb{R}$, $A = [0, 2]$, $B = [1, 3]$, $\bar{A} \setminus \bar{B} = [0, 1)$, $\overline{A \setminus B} = [0, 1]$.
(E^c means the complement of E in X).

3\. $f(0) = f(0+0) = f(0) + f(0)$, so $f(0) = 0$. For $\forall m \in \mathbb{N}$, $f(mx) = f(x + \cdots + x) = f(x) + \cdots + f(x) = mf(x)$. Replace x by $\frac{x}{m}$, we have $f(\frac{1}{m}x) = \frac{1}{m} f(x)$. And $f(0) = f(x - x) =$
$= f(x + (-x)) = f(x) + f(-x)$, so $f(-x) = -f(x)$. Then $f(qx) = qf(x)$ for $\forall q \in \mathbb{Q}$. Moreover $f(q) = f(1)q$, $\forall q \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, any real number is a limit of a sequence of rational numbers. f is continuous, so we have $f(x) = f(1)x$, $x \in \mathbb{R}$. Since $f(1) \in \mathbb{R}$, all these kind of functions are in the form $f(x) = rx$, $r \in \mathbb{R}$.

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4. Let $f: X \to Y$, $g: Y \to Z$ be continuous functions. U is an arbitrary open set in Z. g is continuous, so $g^{-1}(U)$ is open in Y. Similarly, $f^{-1}(g^{-1}(U))$ is open in X. So $f^{-1}(g^{-1}(U)) = f^{-1} \circ g^{-1}(U) = (g \circ f)^{-1}(U)$ is open, $g \circ f$ is continuous

5. Let $X = \mathbb{R}^2$, d be the usual Euclidean metric. Let A be the curve $\{(x,y) \mid x, y > 0,\ xy = 1\}$, B be the x-axis. Then A, B are distinct closed sets. $\operatorname{dist}(A,B) = \inf\{d(u,v) \mid u \in A, v \in B\}$ is zero.
(A is closed since function $y = \frac{1}{x}$ is continuous on $(0, \infty)$ and then $A' \subseteq A$. It's obvious that B is closed).