Introduction to the p-Laplacian

$$1 < p \leq \infty.$$

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1. Introduction

For $p > 1$ consider the p-Laplace equation

$$-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \tag{1}$$

where $u\colon \Omega \mapsto \mathbb{R}$ is a real function defined on a domain $\Omega \subset \mathbb{R}^n$. Equation (1) is the Euler-Lagrange equation of the p-Dirichlet integral

$$\frac{1}{p}\int_\Omega |\nabla u|^p \, dx.$$

For $p = 2$ we just get the usual Laplacian.

For $p > 2$ equation (1) is *degenerate elliptic* and for $1 < p < 2$ *singular*, at points where $\nabla u = 0$.

2. Sobolev Weak Solutions

Multiply equation (1) by a function $\phi \in C_0^\infty(\Omega)$ and integrate by parts to obtain

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = 0. \tag{2}$$

For the integrand to be in L^1 one would need *a priori* to know only that $\nabla u \in L^{p-1}_{\mathrm{loc}}(\Omega)$. We could say that a function in the Sobolev space $W^{1,p-1}_{\mathrm{loc}}(\Omega)$ is a weak solution of equation (1), if (2) holds for every $\phi \in C_0^\infty(\Omega)$.

However, little is known about this class of “ultra” weak solutions. In order to get the first Cacciopoli type estimates it is necessary to use test functions of the form $\eta^p u$ where $\eta \in C_0^\infty(\Omega)$. One needs to assume a priori that $\nabla u \in L^p_{\mathrm{loc}}(\Omega)$.

Definition: A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a (Sobolev) weak solution of the p-Laplace equation if (2) holds for every $\phi \in C_0^\infty(\Omega)$.

Weak solutions of the p-Laplace equation are often called p-*harmonic* functions.

Regularity: Ural'tseva (68) proved that for $p > 2$ weak solutions of equation (1) have Hölder continuous derivatives. Uhlenbeck (75) proved a far reaching extension to elliptic complexes. Lewis (83) and DiBenedetto (83) gave proofs valid for the case $1 < p < 2$. However, in general, solutions do not have any better regularity than $C^{1,\alpha}_{\mathrm{loc}}$.

Sharp regularity in two dimensions: Aronsson (89) and Iwaniec-Manfredi (89) proved that a p-harmonic function is in $C^{k,\alpha}_{\mathrm{loc}}$, where

$$k + \alpha = \frac{1}{6}\left(7 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\,\right)$$

Brief sketch of the proof in $\mathbb{R}^n$:

STEP 1: Approximation

Let $\epsilon > 0$ and let u_ϵ be a solution to the equation

$$-\Delta_{p,\epsilon} u_\epsilon = -\operatorname{div}\left(\left(|\nabla u_\epsilon|^2 + \epsilon^2\right)^{\frac{p-2}{2}} \nabla u_\epsilon\right) = 0 \tag{3}$$

with $u - u_\epsilon \in W_0^{1,p}(\Omega)$. Equation (3) is no longer degenerate elliptic. It follows that $u_\epsilon \in C^\infty_{\text{loc}}(\Omega)$. It turns out that $u_\epsilon \to u$ in $W^{1,p}(\Omega)$ as $\epsilon \to 0$.

Conclusion: If we prove estimates for u_ϵ with constants independent of ϵ, we can let $\epsilon \to 0$ to get estimates for u.

STEP 2: ∇u is locally bounded

Set $\omega = |\nabla u|^p$.

Differentiating (3) with respect to x_i and adding in $i = 1 \ldots n$ one proves that ω is a subsolution of a linear ellipitc equation in divergence form with measurable coefficients. Thus ω is locally bounded.

STEP 3: ∇u is Hölder continuous

There are basically two proofs of this fact, both complicated.

The method of the alternative

(Ural'tseva, Evans, DiBenedetto; see DiBenedetto's book *Degenerate Parabolic Equations*):

Fix $\delta > 0$ and a ball $B_R(x_0)$. If the set of points satisfying $|\nabla u(x)| > \delta|\nabla u(x_0)|$ has non-trivial measure (relative to $B_R(x_0)$), then $|\nabla u(x_0)| > (\delta/2)|\nabla u(x_0)|$ in a ball $B_{\eta R}(x_0)$, $0 < \eta < 1$. In $B_{\eta R}(x_0)$ we get good estimates because the equation is no longer degenerate. If the above fails for a sequence of radii and δ going to zero, then ∇u is Hölder continuous at x_0. One then has to patch these two alternatives together.

Using Gehring's higher integrabilty Lemma

(Lewis; see Guisti's book *Direct Methods in the Calculus of Variations*):

Lewis showed that $v_i = |\nabla u|^{(p-2)/2} \partial_{x_i} u$ solves

$$\mathcal{L} v_i = g_i \geq 0,$$

where $\mathcal{L}$ is is elliptic divergence form with measurable coefficients and $g_i \in L^{1+\sigma}$.

The lack of classical second derivatives prevents the pointwise interpretation of (1) as well as rigorous calculations with second derivatives that formally make sense. The consideration of viscosity solutions of degenerate elliptic equations like (1) provides us with a device to overcome this difficulty.

As in the linear theory ($p = 2$), sub and supersolutions are necessary for the treatment of the obstacle problem and for Perron's method.

Definition A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a (Sobolev) p-supersolution of equation (1) if

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle \, dx \geq 0 \tag{4}$$

for every nonnegative test function $\phi \in C_0^\infty(\Omega)$.

Theorem (Serrin, 64) Every p-supersolution is locally essentially bounded below and it always has a representative that is lower semi-continuous.

Potential Theoretic Weak Solutions

p-supersolutions always satisfy the comparison principle with respect to p-harmonic functions. This property is used to define supersolutions in the potential theoretic sense.

Definition: A lower semi-continuous function $u \colon \Omega \mapsto \mathbb{R} \cup \{+\infty\}$ that is not identically $+\infty$ is p-superharmonic, if it satisfies the comparison principle with respect to p-harmonic functions in every subdomain D with closure in Ω: If a p-harmonic function $h \in C(\overline{D})$ is such that

$$u(x) \geq h(x) \text{ for all } x \in \partial D$$

then

$$u(x) \geq h(x) \text{ for all } x \in D.$$

Theorem (Lindqvist, 86) Every p-supersolution has a lower semicontinuous representative that is p-superharmonic.

Example: The *fundamental solution* given by

$$x \mapsto |x|^{\frac{p-n}{p-1}}$$

for $1 < p < n$ and by

$$x \mapsto \log\left(\frac{1}{|x|}\right)$$

for $p = n$, is p-superharmonic, yet not a p-supersolution in any domain containing the origin.

Theorem (Lindqvist, 86) If v is locally bounded and p-superharmonic, then $v \in W^{1,p}_{\mathrm{loc}}$ and it is a (Sobolev) p-supersolutions.

4. Viscosity Solutions

Local Definition: A lower semi-continuous function $u \colon \Omega \mapsto \mathbb{R} \cup \{+\infty\}$ that is not identically $+\infty$ is a p-supersolution in the viscosity sense if for every $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ touching u from below at x_0, that is

$$
\begin{aligned}
&(i) \quad \phi(x_0) = u(x_0),\\
&(ii) \quad \phi(x) < u(x) \text{ for } x \neq x_0, \text{ and}\\
&(iii)\ \nabla\phi(x_0) \neq 0,
\end{aligned}
\tag{5}
$$

we have

$$
-\operatorname{div}\left(|\nabla\phi|^{p-2}\nabla\phi\right)(x_0) \geq 0. \tag{6}
$$

Note the need for condition (5)(iii) in the pointwise evaluation of (6) in the case $1 < p < 2$, since we need the function $x \mapsto -\operatorname{div}\left(|\nabla\phi|^{p-2}\nabla\phi\right)(x)$ to be defined at every point near x_0.

Remarks: (i) we need only to ask that (5)(ii) holds in a neighborhood of the point x_0,

(ii) by adding $-\epsilon|x-x_0|^4$ to ϕ we can replace "$<$" by "$\leq$ " in (5)(ii) and,

(iii) it suffices to test with quadratic polynomials ϕ.

Definition based on Comparison A lower semi-continuous function $u\colon \Omega \mapsto \mathbb{R}\cup\{+\infty\}$ that is not identically $+\infty$ is a p-supersolution in the viscosity sense, if for every domain D whose closure is contained in Ω and for every $\phi \in C^2(D) \cap C(\overline{D})$ such that

$$\begin{cases} -\operatorname{div}\left(|\nabla\phi|^{p-2}\nabla\phi\right) & < 0 \text{ in } D \\ \phi & \le u \text{ on } \partial D \end{cases}$$

we have $\phi \le u$ in D.

Lemma 1: Local Definition $\equiv$ Definition based on comparison.

Lemma 2: Every p-superharmonic function is a p-supersolution in the viscosity sense.

We have three different notions of weak supersolutions in increasing order of generality:

p-supersolutions,

p-superharmonic functions, and

p-supersolutions in the viscosity sense.

The relationship between the first two is very well understood. Locally bounded p-superharmonic functions are p-supersolutions and a given p-superharmonic function is is a monotone increasing pointwise limit of p-supersolutions.

Theorem 1 (Juutinen-Lindqvist-M, 01)
p-superharmonic functions = p-supersolutions in the viscosity sense.

In order to prove this theorem, we must show that p-supersolutions in the viscosity sense satisfy the comparison principle with respect to p-harmonic functions. If one knew that p-harmonic functions could be approximated by C^2-smooth strict supersolutions, the converse would follow easily. However, such an approximation result is not known to us for $p \neq 2$.

Theorem 2 (Juutinen-Lindqvist-M, 01) Suppose that u is a p-subsolution in viscosity sense and v is a p-supersolution in the viscosity sense in a bounded domain Ω. If for all $x \in \partial\Omega$ we have

$$\limsup_{y \to x} u(y) \leq \liminf_{y \to x} v(y)$$

and both sides are not simultaneously ∞ or $-\infty$, then $u(x) \leq v(x)$ for all $x \in \Omega$.

The proof of this theorem is based on the *maximum principle for semi-continuous functions* of Crandall-Ishii-Lions-Jensen (92).

Jets

Definition: Let v be an extended real valued function defined in a domain Ω. For a point $x_0 \in \Omega$ we define the second order sub-jet $J^{2,-}(v, x_0)$ as the set of all pairs $(\eta, X) \in \mathbb{R}^n \times \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the set of symmetric $n \times n$ real matrices, such that as $x \to x_0$ we have

$$v(x) \geq v(x_0) + \langle \eta, x - x_0 \rangle + \frac{1}{2}\langle X(x - x_0), x - x_0 \rangle + o\left(|x - x_0|^2\right).$$

Definition: Let u be an extended real valued function defined in a domain Ω. For $x_0 \in \Omega$ we define the second order super-jet $J^{2,+}(u, x_0)$ as the set of all pairs $(\eta, X) \in \mathbb{R}^n \times \mathcal{S}(\mathbb{R}^n)$ such that as $x \to x_0$ we have

$$u(x) \leq u(x_0) + \langle \eta, x - x_0 \rangle + \frac{1}{2}\langle X(x - x_0), x - x_0 \rangle + o\left(|x - x_0|^2\right).$$

Facts about Jets:

(i) the sets $J^{2,+}(u,x)$ and $J^{2,-}(u,x)$ could very well be empty.

(ii) If $J^{2,+}(u,x) \cap J^{2,-}(u,x) \neq \emptyset$, then it contains only one pair (η_0, X_0). Moreover, the function u is differentiable at x_0, the vector $\eta_0 = \nabla u(x_0)$ and we say that u is twice pointwise differentiable at x_0 and write $D^2u(x_0) = X_0$.

(iii) Jets are determined by smooth functions ϕ that touch a function u from above or below at a point $x_0 \in \Omega$. Denote by $K^{2,-}(u, x_0)$ the collection of pairs

$$\Big(\nabla\phi(x_0), D^2\phi(x_0)\Big) \in \mathbb{R}^n \times \mathcal{S}(\mathbb{R}^n)$$

where $\phi \in C^2(\Omega)$ touches u from below at x_0; that is, $\phi(x_0) = u(x_0)$ and $\phi(x) < u(x)$ for $x \neq x_0$. Similarly, we define $K^{2,+}(u, x_0)$ using smooth test functions that touch a function u from above. In fact we have:

Lemma (Ishii-Crandall, 96):

$$K^{2,+}(u, x_0) = J^{2,+}(u, x_0)$$

and

$$K^{2,-}(u, x_0) = J^{2,-}(u, x_0).$$

From this lemma we see that the local definition and the definition based on comparison of viscosity supersolutions are equivalent to:

Jets Definition: A lower semi-continuous function $u\colon \Omega \mapsto \mathbb{R}\cup\{+\infty\}$ that is not identically $+\infty$ is a p-supersolution in the viscosity sense, if for every $x_0 \in \Omega$ and every pair $(\eta, X) \in J^{2,-}(u, x_0)$ with $\eta \neq 0$, we have

$$-\Big[|\eta|^{p-2}\,\mathrm{trace}(X) + (p-2)|\eta|^{p-4}\langle X\cdot\eta, \eta\rangle\Big] \geq 0. \tag{7}$$

Note that (7) can be replaced by

$$-\Big[|\eta|^{2}\,\mathrm{trace}(X) + (p-2)\langle X\cdot\eta, \eta\rangle\Big] \geq 0 \tag{8}$$

without affecting the notion of p-supersolution.

∞-harmonic functions

What is the limit of the p-Laplacian as $p \to \infty$? Let u_p be the solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) & = 0 \text{ in } \Omega \\ u_p & = F \text{ on } \partial\Omega. \end{cases} \tag{9}$$

where the domain Ω and the boundary datum F are smooth. Does the limit of u_p exist as $p \to \infty$? If so, what equation does it satisfy?

To discover the equation that u_∞ must satisfy, let us proceed formally and divide (8) by $p-2$ and let $p \to \infty$. We obtain that for every pair $(\eta, X) \in J^{2,-}(u_\infty, x_0)$ we must have

$$-\langle X \cdot \eta, \eta \rangle \geq 0.$$

This argument can be made rigorous (by using jets) to conclude that u_∞ is a viscosity solution of the equation

$$-\Delta_\infty u = -\langle D^2 u \cdot \nabla u, \nabla u \rangle = 0 \tag{10}$$

in Ω. The operator on the left-hand side of (10) is denoted Δ_∞ and is given by

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

It is not clear whether notions of weak solution other than viscosity solutions apply in this case. Naturally, this operator is called the ∞-Laplacian and the solutions of the equation $-\Delta_\infty u = 0$ are called ∞-harmonic functions.

For a finite p, the unique solution to (9) minimizes the p-Dirichlet integral

$$\frac{1}{p}\int_\Omega |\nabla u|^p \, dx$$

among all functions with boundary values F. Letting $p \to \infty$ one would guess that u_∞ minimizes the sup-norm of the gradient among all functions with boundary values F. This is, indeed, the case. Moreover, this minimization property still holds when restricting u_∞ to any subdomain of Ω (Aronsson, 67)

We could say that (10) is the Euler-Lagrange equation of the functional $\|\nabla u\|_\infty$.

So far we have indicated how to show the existence of ∞-harmonic functions with given boundary values.

Jensen (93) established uniqueness in the viscosity class, thereby showing that the Dirichlet problem for $-\Delta_\infty$ is well posed.

Eigenvalue problems

Up to multiplication by a positive constant there exists a unique positive function $u_p \in W_0^{1,p}(\Omega)$ that minimizes the p-Rayleigh quotient

$$J_p(u) = \frac{\left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p}}{\left(\int_\Omega |u|^p \, dx\right)^{1/p}}$$

among all nonzero functions $u \in W_0^{1,p}(\Omega)$.

Let Λ_p be the minimum of J_p. Then the p-ground state u_p is a solution of the equation

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \Lambda_p^p |u|^{p-2} u. \tag{11}$$

We ask now what should be the equation that the ∞-ground state satisfies. This number turns out to be the reciprocal of the radius of the largest ball in Ω

$$\Lambda_\infty = \frac{1}{\max\{d(x,\partial\Omega)\colon x \in \Omega\}}.$$

One can now proceed formally to obtain that u_∞ must be a solution of the equation

$$\min\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0. \tag{12}$$

This calculation can indeed be made rigorous (Juutinen-Lindqvist-M, Fukagai-Ito-Narukawa, 99).

In the case of a ball, it is known that the distance to the boundary is an ∞-ground state, since it is the limit of p-ground states. For more complicated geometries, we can use the equation for the ∞-ground states to prove that this is not the case. For example, when Ω is a square, the distance to the boundary $d(x, \partial\Omega)$ is not an ∞-ground state, although it minimizes the formal limit of the functionals J_p as $p \to \infty$,

$$J_\infty(u) = \frac{\|\nabla u\|_\infty}{\|u\|_\infty}.$$

To obtain deeper results we must study the uniqueness of ∞-ground states. So far as we know, uniqueness has only been established in the case when Ω is a ball, where the only solution is the distance to the boundary. However, we do have uniqueness for the Dirichlet problem for the equation (12) if the boundary datum is strictly positive.

Corollary: If we have a non-trivial solution to (12) with any $\Lambda \in \mathbb{R}$ in place of Λ_∞, then indeed $\Lambda = \Lambda_\infty$.

Embarrassingly Simple Open Problems

Remember $p \neq 2$.

Unique Continuation: Let $n \geq 3$ and u be a p-harmonic function in $B_{2R} \subset \mathbb{R}^n$ such that $u \equiv 0$ in the ball B_R. Is $u \equiv 0$ in B_{2R}?

Strong Comparison Principle: Let $n \geq 3$ and u and v be a p-harmonic functions in $B_R(0) \subset \mathbb{R}^n$ such that $u(x) \leq v(x)$ for all $x \in B_R(0)$ and $u(0) = v(0)$. Is $u \equiv v$ in $B_R(0)$?

Boundary Comparison Principle: Let $\mathbb{D}$ denote the unit disk in $\mathbb{R}^2$. For $\delta > 0$ consider I_δ the arc centered at $(1,0)$ with length $\delta/2$. Given $\varepsilon > 0$ find $\delta > 0$ depending only on ε, M, and p such that

$$|u(0) - v(0)| \leq \varepsilon$$

for all p-harmonic functions u and v in $\mathbb{D}$ that extend smoothly to $\overline{\mathbb{D}}$, are bounded $\|u\|_{L^\infty(\mathbb{D})} \leq M$, $\|u\|_{L^\infty(\mathbb{D})} \leq M$, and satisfy $u(y) = v(y)$ for all $y \in \partial\mathbb{D} \setminus I_\delta$

p-Harmonic Measure Estimates: Let $\mathbb{D}$ denote the unit disk in $\mathbb{R}^2$ and the arc I_δ defined as before. How does the p-harmonic *measure* of I_δ behaves as $\delta \to 0$? Is there a number α such that

$$\omega_p(I_\delta, 0, \mathbb{D}) \sim \delta^\alpha$$

as $\delta \to 0$?