k-point configurations in vector spaces over finite fields

ABSTRACT

The classical Erdős distance problem asks for the minimal number of distinct distances determined by a finite point set in $\mathbb{R}^d$ where $d \geq 2$. More precisely, the problem is to find the smallest possible size of $\Delta(E)$ in terms of the size of E where $\Delta(E) = \{||x - y|| : x, y \in E\}$ and $E \subset \mathbb{R}^d$ is finite. The Erdős conjecture is that $|\Delta(E)| \gtrapprox |E|^{2/d}$. The continuous analog of this problem, called the Falconer distance problem, is to find $s_0 > 0$ such that if the Hausdorff dimension of E is greater than s_0, then the Lebesgue measure of $\Delta(E)$ is positive. It is conjectured that the Lebesgue measure of $\Delta(E)$ is positive provided that the Hausdorff dimension of E is greater than $\frac{d}{2}$.

In vector spaces over finite fields, one may define $\Delta(E) = \{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2 : x, y \in E\}$, and one may again ask for the smallest possible size of $\Delta(E)$ in terms of the size of E. The first non-trivial result on the Erdős-Falconer distance problem in vector spaces over finite fields was proved by Bourgain, Katz and Tao (Geom. Funct. Anal. **14** (2004), 27-57). They consider the case $d = 2$ and prove that if q is a prime $\equiv 3 \ (mod\ 4)$ and $|E| \lesssim q^{2-\epsilon}$ for some $\epsilon > 0$, then $|\Delta(E)| \gtrsim |E|^{\frac{1}{2}+\delta}$, where δ is a function of ϵ.

A generalization of the Erdős-Falconer distance problem in vector spaces over finite fields is to determine the minimal $\alpha > 0$ such that E contains a congruent copy of every k dimensional simplex, that is $k+1$ points spanning a k-dimensional subspace, whenever $|E| \gtrsim q^{\alpha}$. We improve on known results (for $k > 3$) using Fourier analytic methods, showing that α may be taken to be $\frac{d+k}{2}$.