Design Example*

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1 Introduction

The MATLAB programs listed here reproduce Example 1 in the paper

> I. W. Selesnick and A. Farras Abdelnour. Symmetric wavelet tight frames with two generators. *Applied and Computational Harmonic Analysis*, 17(2):211-225, September 2004. (Special Issue: Frames in Harmonic Analysis, Part II.)

The next pages are obtained by running the program `DesignExample.m`. This uses subprograms listed after.

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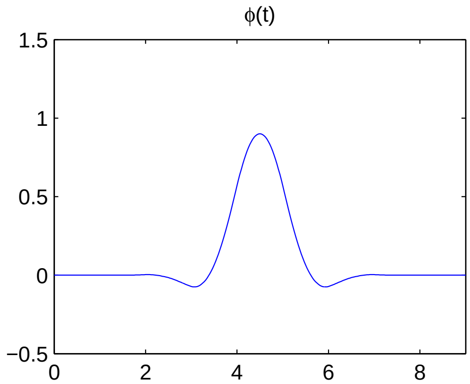
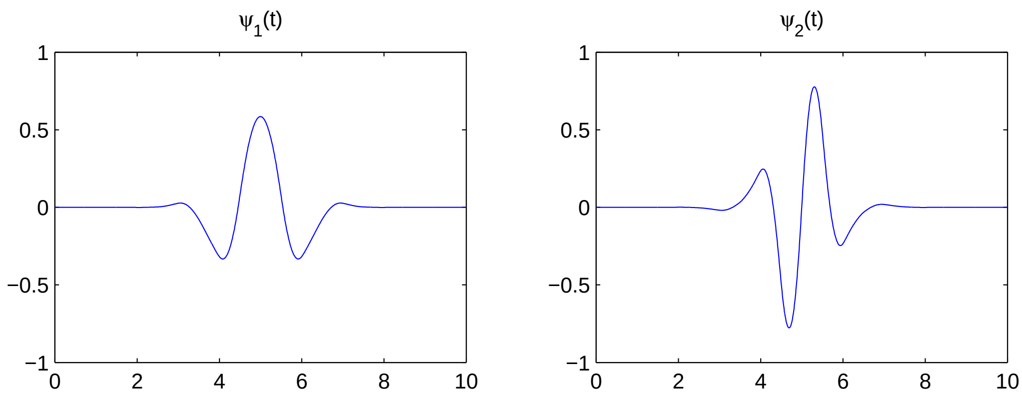
The scaling function and wavelets





Filter Design for Symmetric Wavelet Tight Frames with Two Generators

This program reproduces the design in Example 1 of the paper: I. W. Selesnick and A. Farras Abdelnour, Symmetric wavelet tight frames with two generators, Applied and Computational Harmonic Analaysis, 17(2), 2004.

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Contents

Find alpha

Find the scaling filter h0 as a linear comintation of two symmetric maximally-flat filters.

```
format
K0 = 5;    % K0: H0(z) will have (1+z)^K0 as a factor
VM = 2;    % VM: Number of vanishing moments

% "Symmetric Maximally-Flat" lowpass filters in Equation (46)
f0 = conv([-7 22 -7],binom(7,0:7))/2^10;
f1 = conv([0 -5 18 -5 0],binom(5,0:5))/2^8;

% Compute G0(z) and G1(z) in Equations (48) and (49)
g0 = sqrt(2)*f0(1:2:end);
g1 = sqrt(2)*f1(1:2:end);

% Compute the terms in Equation (50)
r0 = conv(g0,flip(g0));        % G0(z)G0(1/z)
r1 = conv(g1,flip(g1));        % G1(z)G1(1/z)
r01 = conv(g0,flip(g1))+conv(flip(g0),g1);   % G0(z)G1(1/z) + G0(1/z)G1(z)

% Compute the coefficients of alpha in Equation (50)
a = (-4:4==0) - 2*r0;
b = 4*r0 - 2*r01;
c = 2*r01 - 2*r0 - 2*r1;

% Compute the common factor, (-1/z + 2 + z)^2
s = [-1 2 -1]/4;
s = conv(s,s);

% For numerical accuracy, remove the common factor from each term
A = extractf(a,s);
B = extractf(b,s);
C = extractf(c,s);
```

```
% Perform the change of variables x = (-z + 2 - 1/z)/4
% to get the polynomials in x in the equation immediately
% before Equation (51)
q0 = z2x(A);
q1 = z2x(B);
q2 = z2x(C);

% It can be verified (in Maple, for example) that q0, q1, q2
% are exactly the following:
q0 = [-189  84  1008]/2^10;
q1 = [ 238  56  -448]/2^10;
q2 = [ -49   0     0]/2^10;

% Rearrange the coefficients to get P0(alpha), P1(alpha),
% P2(alpha) in Equation (51)
P = [q2; q1; q0];
p2 = P(:,1)';
p1 = P(:,2)';
p0 = P(:,3)';

% Compute the discriminant D(alpha)
discrim = conv(p1,p1) - 4*conv(p0,p2);

% The leading coefficient is 0, so let us remove it
discrim = discrim(2:end);

% It can be verified (in Maple, for example) that the
% discriminant is exactly
% discrim = [-112 800 -1644 981]*7^2/2^16
% so for numerical accuracy let us set
discrim = [-112 800  -1644 981];

% Compute the roots of the discriminant
rts = roots(discrim);

% The smallest of the roots gives the smoothest scaling function
alpha = min(rts)
```

```
alpha =

    1.0720
```

Find the scaling filter h0

```
h0 = sqrt(2)*(alpha*f1 + (1-alpha)*f0)

% Compute smoothness coefficients (needs programs by Ojanen, for example)
% M = 2;
% K0 = 5;
% disp('SMOOTHNESS: ')
% sobolev(h0,K0,M)
% sobexp(h0,K0)
% holder(h0,K0,M)
```

```
h0 =

    0.0007   -0.0269   -0.0415    0.1906    0.5842    0.5842    0.1906   -0.0415   -0.0269
0.0007
```

Verify that h0 satisfies Petukhov's condition

The scaling filter H0(z) must satisfy Petukhov's condition: that the roots of 2 - H0(z) H0(1/z) - H0(-z) H0(-1/z) are of even degree, or equivalently, that the roots of 1 - 2 H00(z) H00(1/z) are of even degree.

```
rr = conv(h0,flip(h0));
rr2 = rr; rr(1:2:end) = -rr(1:2:end);
M = (length(rr)-1)/2;

% Find 2 - H0(z) H0(1/z) - H(-z) H(-1/z)
Chk = 2*((-M:M) == 0) - (rr + rr2);
% Verify that all its roots are of even degree
ChkDbleRoots = roots(Chk)

% Find 1 - 2 H00(z) H00(1/z)
h00 = h0(1:2:end);
Chk = ((-4:4) == 0) - 2*conv(h00,flip(h00));
% Verify that all its roots are of even degree
ChkDbleRoots = roots(Chk)
```

```
ChkDbleRoots =

        0
   5.5929
   5.5929
  -5.5929
  -5.5929
   1.0001 + 0.0001i
   1.0001 - 0.0001i
   0.9999 + 0.0001i
   0.9999 - 0.0001i
  -1.0001 + 0.0001i
  -1.0001 - 0.0001i
  -0.9999 + 0.0001i
  -0.9999 - 0.0001i
   0.1788
   0.1788
  -0.1788
  -0.1788


ChkDbleRoots =

  31.2802
  31.2802
   1.0002 + 0.0002i
   1.0002 - 0.0002i
   0.9998 + 0.0002i
   0.9998 - 0.0002i
   0.0320
   0.0320
```

Find the polynomial U(z)

```
% Find the polyphase component H00(z)
h00 = h0(1:2:end);
N = length(h0);
n = 1-N/2:N/2-1;
```

```
% Find roots of H00(z)
rts_h00 = roots(h00);             % Values (52) in paper

% Find U(z) via spectral factorization of 1 - 2 H00(z) H00(1/z)
% Note: u should be a symmetric sequence.

% Find U(z)^2 from using Equation (20)
u2 = (n==0) - 2*conv(h00,flip(h00));

% For numerical accuracy, factor (-1/z + 2 + z)^2 out of U(z)^2
ff = extractf(u2,[1 -4 6 -4 1]);

% Find the roots (use 'dbleroots' function to improve numerical accuracy)
rts_f = dbleroots(ff);

% Form polynomial from the roots
u = poly(rts_f);

% Multiply with (1/z - 2 + z)
u = conv(u, [1 -2 1]);

% Correctly normalize U(z)
u = u*sqrt(u2(1));

% Check that U(z) U(1/z) = 1 - 2 H00(z) H00(1/z)
ChkZeros = conv(u,flip(u)) - u2   % this should be zero
```

```
ChkZeros =

   1.0e-12 *

    0.0000   -0.0036    0.0847   -0.3072    0.4523   -0.3072    0.0847   -0.0036    0.0000
```

Find the polynomials A(z) and B(z)

```
% Find 0.5 + 0.5 U(z) and 0.5 - 0.5 U(z)
n = (1-N/2)/2:(N/2-1)/2;
ra = 0.5*(n==0) + 0.5*u;          % 0.5 + 0.5 U(z)
rb = 0.5*(n==0) - 0.5*u;          % 0.5 - 0.5 U(z)

% Find roots of 0.5 + 0.5 U(z) and 0.5 - 0.5 U(z)
rts_ra = roots(ra);               % Values (53) in paper
rts_rb = roots(rb);               % Values (54) in paper

% Determine the roots of A(z) and B(z) according to paper
rts_a = [];
rts_b = [];
for k = 1:4
    [tmp1,k1] = min(abs(rts_h00(k)-rts_ra));
    [tmp2,k2] = min(abs(rts_h00(k)-rts_rb));
    if tmp1 < tmp2
        rts_a = [rts_a rts_h00(k)];
    else
        rts_b = [rts_b 1/rts_h00(k)];
    end
end

% Find A(z) and B(z)
a = poly(rts_a);
```

```
b = poly(rts_b);
a = a/sum(a)/sqrt(2)   % Normalize A(z) so that A(1) = 1/sqrt(2)
b = b/sum(b)/sqrt(2)   % Normalize B(z) so that B(1) = 1/sqrt(2)
```

```
a =

  -0.0347    0.8300   -0.0881


b =

   0.2160    0.5053   -0.0142
```

Verify that A(z) A(1/z) + B(z) B(1/z) = 1

```
ChkDelta = conv(a,flip(a)) + conv(b,flip(b))   % should be delta(n)
```

```
ChkDelta =

   0.0000   -0.0000    1.0000   -0.0000    0.0000
```

Find the wavelet filters h1 and h2

The filter h2 will be the time-reversed version of h1

```
% Determine H10(z) and H11(z)
h10 = conv(a,a);             % H10(z) = A^2(z)
h11 = -conv(b,b);            % H11(z) = -B^2(z)

% Determine H1(z) and H2(z)
h1 = [h10; h11]; h1 = h1(:)';
h2 = flip(h1);

% Display filter coefficients
format long
Table1 = [h0' h1' h2']               % Table 1 in paper
```

```
Table1 =

   0.00069616789827   0.00120643067872  -0.00020086099895
  -0.02692519074183  -0.04666026144290   0.00776855801988
  -0.04145457368921  -0.05765656504458   0.01432190717031
   0.19056483888762  -0.21828637525088  -0.14630790303599
   0.58422553883170   0.69498947938197  -0.24917440947758
   0.58422553883170  -0.24917440947758   0.69498947938197
   0.19056483888762  -0.14630790303599  -0.21828637525088
  -0.04145457368921   0.01432190717031  -0.05765656504458
  -0.02692519074183   0.00776855801988  -0.04666026144290
   0.00069616789827  -0.00020086099895   0.00120643067872
```

Verify the perfect reconstruction conditions

```
g0 = flip(h0);
g1 = flip(h1);
```

```
g2 = flip(h2);
pr1 = conv(h0,g0) + conv(h1,g1) + conv(h2,g2);
N = length(h0);
s = (-1).^(0:N-1);
pr2 = conv(h0.*s,g0) + conv(h1.*s,g1) + conv(h2.*s,g2);
CheckPR = [pr1' pr2']
```

```
CheckPR =

   0.00000000000000   0.00000000000000
  -0.00000000000000   0.00000000000000
   0.00000000000000  -0.00000000000000
   0.00000000000000   0.00000000000000
  -0.00000000000000   0.00000000000000
   0.00000000000000                  0
   0.00000000000000  -0.00000000000000
  -0.00000000000000  -0.00000000000000
   0.00000000000000  -0.00000000000000
   2.00000000000000  -0.00000000000000
   0.00000000000000   0.00000000000000
  -0.00000000000000  -0.00000000000000
   0.00000000000000   0.00000000000000
   0.00000000000000                  0
  -0.00000000000000  -0.00000000000000
   0.00000000000000   0.00000000000000
   0.00000000000000   0.00000000000000
  -0.00000000000000   0.00000000000000
   0.00000000000000  -0.00000000000000
```

Find (anti-) symmetric wavelet filters h1, h2

The filters h1 and h2 are flips of one another, and neither are symmetric. Let us convert them to a symmetric and an anti-symmetric pair.

```
% Shift h1 by 2 samples
h0 = [h0 0 0];
h1 = [0 0 h1];
h2 = [h2 0 0];

% Replace h1 and h2 by their sum and difference
tmp1 = (h1+h2)/sqrt(2);
tmp2 = (h1-h2)/sqrt(2);
h1 = tmp1;
h2 = tmp2;

% Display filter coefficients
Table2 = [h0' h1' h2']          % Table 2 in paper
```

```
Table2 =

   0.00069616789827  -0.00014203017443   0.00014203017443
  -0.02692519074183   0.00549320005590  -0.00549320005590
  -0.04145457368921   0.01098019299360  -0.00927404236569
   0.19056483888762  -0.13644909765614   0.07046152309972
   0.58422553883170  -0.21696226276270   0.13542356651680
   0.58422553883170   0.33707999754377  -0.64578354990483
   0.19056483888762   0.33707999754377   0.64578354990483
  -0.04145457368921  -0.21696226276270  -0.13542356651680
  -0.02692519074183  -0.13644909765614  -0.07046152309972
```

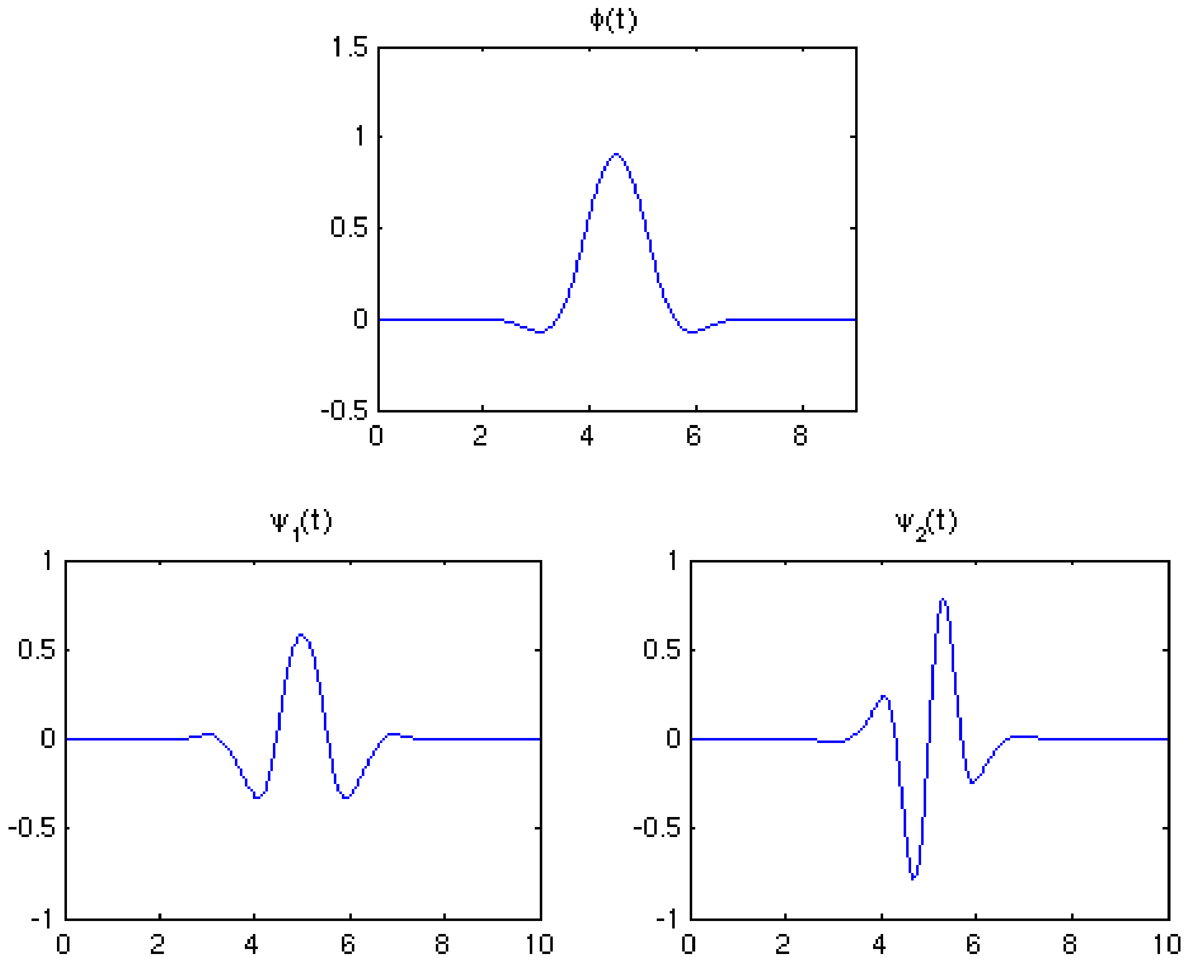
```
0.00069616789827   0.01098019299360   0.00927404236569
               0   0.00549320005590   0.00549320005590
               0  -0.00014203017443  -0.00014203017443
```

Plot the scaling function and wavelets

```
h0 = h0(1:10);
[s0,t0] = scalfn(h0);           % Compute the scaling function
[w1,s,t] = wletfn(h0,h1);       % Compute the first wavelet
[w2,s,t] = wletfn(h0,h2);       % Compute the second wavelet

figure(1)
s1 = subplot(2,2,1);
ax1 = get(s1,'position');
s2 = subplot(2,2,2);
ax2 = get(s2,'position');
clf
s3 = subplot(2,2,1);
set(s3,'position',(ax1+ax2)/2);
plot(t0,s0);
title('\phi(t)')
axis([0 9 -0.5 1.5])
%
subplot(2,2,3)
plot(t,w1);
axis([0 10 -1 1])
title('\psi_1(t)')
%
subplot(2,2,4)
plot(t,w2)
axis([0 10 -1 1])
title('\psi_2(t)')
print -dpsc plots


% Make Figure 2 in paper (phase plot)
if 0
    [BA,w] = freqz(b,a);
    figure(2)
    subplot(2,2,1)
    plot(w/pi,angle(BA)/pi,w/pi,0.25*w/pi,':')
    xlabel('\omega/\pi')
    title('[\angle{B(e^{j \omega})/A(e^{j \omega})}]/\pi')
    axis square
    % print -dps phase
end
```



φ(t)
ψ1(t)
ψ2(t)

Program Listing

DesignExample.m

```
%% Filter Design for Symmetric Wavelet Tight Frames with Two Generators
% This program reproduces the design in Example 1 of the paper:
% I. W. Selesnick and A. Farras Abdelnour,
% Symmetric wavelet tight frames with two generators,
% Applied and Computational Harmonic Analalysis, 17(2), 2004.
%
% Ivan Selesnick, selesi@poly.edu, Polytechnic University, Brooklyn, NY

%% Find alpha
% Find the scaling filter h0 as a linear comintation of two
% symmetric maximally-flat filters.

format
K0 = 5;    % K0: H0(z) will have (1+z)^K0 as a factor
VM = 2;    % VM: Number of vanishing moments

% "Symmetric Maximally-Flat" lowpass filters in Equation (46)
f0 = conv([-7 22 -7],binom(7,0:7))/2^10;
f1 = conv([0 -5 18 -5 0],binom(5,0:5))/2^8;

% Compute G0(z) and G1(z) in Equations (48) and (49)
g0 = sqrt(2)*f0(1:2:end);
g1 = sqrt(2)*f1(1:2:end);

% Compute the terms in Equation (50)
r0 = conv(g0,flip(g0));        % G0(z)G0(1/z)
r1 = conv(g1,flip(g1));        % G1(z)G1(1/z)
r01 = conv(g0,flip(g1))+conv(flip(g0),g1);   % G0(z)G1(1/z) + G0(1/z)G1(z)

% Compute the coefficients of alpha in Equation (50)
a = (-4:4==0) - 2*r0;
b = 4*r0 - 2*r01;
c = 2*r01 - 2*r0 - 2*r1;

% Compute the common factor, (-1/z + 2 + z)^2
s = [-1 2 -1]/4;
s = conv(s,s);

% For numerical accuracy, remove the common factor from each term
A = extractf(a,s);
```

```
B = extractf(b,s);
C = extractf(c,s);

% Perform the change of variables x = (-z + 2 - 1/z)/4
% to get the polynomials in x in the equation immediately
% before Equation (51)
q0 = z2x(A);
q1 = z2x(B);
q2 = z2x(C);

% It can be verified (in Maple, for example) that q0, q1, q2
% are exactly the following:
q0 = [-189  84  1008]/2^10;
q1 = [ 238  56  -448]/2^10;
q2 = [ -49   0     0]/2^10;

% Rearrange the coefficients to get P0(alpha), P1(alpha),
% P2(alpha) in Equation (51)
P = [q2; q1; q0];
p2 = P(:,1)';
p1 = P(:,2)';
p0 = P(:,3)';

% Compute the discriminant D(alpha)
discrim = conv(p1,p1) - 4*conv(p0,p2);

% The leading coefficient is 0, so let us remove it
discrim = discrim(2:end);

% It can be verified (in Maple, for example) that the
% discriminant is exactly
% discrim = [-112 800 -1644 981]*7^2/2^16
% so for numerical accuracy let us set
discrim = [-112 800  -1644 981];

% Compute the roots of the discriminant
rts = roots(discrim);

% The smallest of the roots gives the smoothest scaling function
alpha = min(rts)

%% Find the scaling filter h0
```

```
h0 = sqrt(2)*(alpha*f1 + (1-alpha)*f0)

% Compute smoothness coefficients (needs programs by Ojanen, for example)
% M = 2;
% K0 = 5;
% disp('SMOOTHNESS: ')
% sobolev(h0,K0,M)
% sobexp(h0,K0)
% holder(h0,K0,M)

%% Verify that h0 satisfies Petukhov's condition
% The scaling filter H0(z) must satisfy Petukhov's condition:
% that the roots of 2 - H0(z) H0(1/z) - H0(-z) H0(-1/z) are of
% even degree, or equivalently, that the roots of
% 1 - 2 H00(z) H00(1/z) are of even degree.

rr = conv(h0,flip(h0));
rr2 = rr; rr(1:2:end) = -rr(1:2:end);
M = (length(rr)-1)/2;

% Find 2 - H0(z) H0(1/z) - H(-z) H(-1/z)
Chk = 2*((-M:M) == 0) - (rr + rr2);
% Verify that all its roots are of even degree
ChkDbleRoots = roots(Chk)

% Find 1 - 2 H00(z) H00(1/z)
h00 = h0(1:2:end);
Chk = ((-4:4) == 0) - 2*conv(h00,flip(h00));
% Verify that all its roots are of even degree
ChkDbleRoots = roots(Chk)

%% Find the polynomial U(z)

% Find the polyphase component H00(z)
h00 = h0(1:2:end);
N = length(h0);
n = 1-N/2:N/2-1;

% Find roots of H00(z)
rts_h00 = roots(h00);            % Values (52) in paper

% Find U(z) via spectral factorization of 1 - 2 H00(z) H00(1/z)
```

```
% Note: u should be a symmetric sequence.

% Find U(z)^2 from using Equation (20)
u2 = (n==0) - 2*conv(h00,flip(h00));

% For numerical accuracy, factor (-1/z + 2 + z)^2 out of U(z)^2
ff = extractf(u2,[1 -4 6 -4 1]);

% Find the roots (use 'dbleroots' function to improve numerical accuracy)
rts_f = dbleroots(ff);

% Form polynomial from the roots
u = poly(rts_f);

% Multiply with (1/z - 2 + z)
u = conv(u, [1 -2 1]);

% Correctly normalize U(z)
u = u*sqrt(u2(1));

% Check that U(z) U(1/z) = 1 - 2 H00(z) H00(1/z)
ChkZeros = conv(u,flip(u)) - u2   % this should be zero

%% Find the polynomials A(z) and B(z)

% Find 0.5 + 0.5 U(z) and 0.5 - 0.5 U(z)
n = (1-N/2)/2:(N/2-1)/2;
ra = 0.5*(n==0) + 0.5*u;          % 0.5 + 0.5 U(z)
rb = 0.5*(n==0) - 0.5*u;          % 0.5 - 0.5 U(z)

% Find roots of 0.5 + 0.5 U(z) and 0.5 - 0.5 U(z)
rts_ra = roots(ra);               % Values (53) in paper
rts_rb = roots(rb);               % Values (54) in paper

% Determine the roots of A(z) and B(z) according to paper
rts_a = [];
rts_b = [];
for k = 1:4
    [tmp1,k1] = min(abs(rts_h00(k)-rts_ra));
    [tmp2,k2] = min(abs(rts_h00(k)-rts_rb));
    if tmp1 < tmp2
        rts_a = [rts_a rts_h00(k)];
```

```
    else
        rts_b = [rts_b 1/rts_h00(k)];
    end
end

% Find A(z) and B(z)
a = poly(rts_a);
b = poly(rts_b);
a = a/sum(a)/sqrt(2)   % Normalize A(z) so that A(1) = 1/sqrt(2)
b = b/sum(b)/sqrt(2)   % Normalize B(z) so that B(1) = 1/sqrt(2)

%% Verify that A(z) A(1/z) + B(z) B(1/z) = 1
ChkDelta = conv(a,flip(a)) + conv(b,flip(b))   % should be delta(n)


%% Find the wavelet filters h1 and h2
% The filter h2 will be the time-reversed version of h1

% Determine H10(z) and H11(z)
h10 = conv(a,a);              % H10(z) = A^2(z)
h11 = -conv(b,b);             % H11(z) = -B^2(z)

% Determine H1(z) and H2(z)
h1 = [h10; h11]; h1 = h1(:)';
h2 = flip(h1);

% Display filter coefficients
format long
Table1 = [h0' h1' h2']                % Table 1 in paper

%% Verify the perfect reconstruction conditions

g0 = flip(h0);
g1 = flip(h1);
g2 = flip(h2);
pr1 = conv(h0,g0) + conv(h1,g1) + conv(h2,g2);
N = length(h0);
s = (-1).^(0:N-1);
pr2 = conv(h0.*s,g0) + conv(h1.*s,g1) + conv(h2.*s,g2);
CheckPR = [pr1' pr2']

%% Find (anti-) symmetric wavelet filters h1, h2
```

```
% The filters h1 and h2 are flips of one another, and neither are symmetric.
% Let us convert them to a symmetric and an anti-symmetric pair.

% Shift h1 by 2 samples
h0 = [h0 0 0];
h1 = [0 0 h1];
h2 = [h2 0 0];

% Replace h1 and h2 by their sum and difference
tmp1 = (h1+h2)/sqrt(2);
tmp2 = (h1-h2)/sqrt(2);
h1 = tmp1;
h2 = tmp2;

% Display filter coefficients
Table2 = [h0' h1' h2']          % Table 2 in paper


%% Plot the scaling function and wavelets

h0 = h0(1:10);
[s0,t0] = scalfn(h0);           % Compute the scaling function
[w1,s,t] = wletfn(h0,h1);       % Compute the first wavelet
[w2,s,t] = wletfn(h0,h2);       % Compute the second wavelet

figure(1)
s1 = subplot(2,2,1);
ax1 = get(s1,'position');
s2 = subplot(2,2,2);
ax2 = get(s2,'position');
clf
s3 = subplot(2,2,1);
set(s3,'position',(ax1+ax2)/2);
plot(t0,s0);
title('\phi(t)')
axis([0 9 -0.5 1.5])
%
subplot(2,2,3)
plot(t,w1);
axis([0 10 -1 1])
title('\psi_1(t)')
%
```

```
subplot(2,2,4)
plot(t,w2)
axis([0 10 -1 1])
title('\psi_2(t)')
print -depsc plots


% Make Figure 2 in paper (phase plot)
if 0
    [BA,w] = freqz(b,a);
    figure(2)
    subplot(2,2,1)
    plot(w/pi,angle(BA)/pi,w/pi,0.25*w/pi,':')
    xlabel('\omega/\pi')
    title('[\angle{B(e^{j \omega})/A(e^{j \omega})}]/\pi')
    axis square
    % print -deps phase
end
```

binom.m

```
function a = binom(n,k)
%
% a = binom(n,k)
% BINOMIAL COEFFICIENTS
%
% allowable inputs:
%        n : integer, k : integer
%        n : integer vector, k : integer
%        n : integer, k : integer vector
%        n : integer vector, k : integer vector (of equal dimension)
%

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nv = n;
kv = k;
if (length(nv) == 1) & (length(kv) > 1)
        nv = nv * ones(size(kv));
elseif (length(nv) > 1) & (length(kv) == 1)
        kv = kv * ones(size(nv));
end
a = nv;
for i = 1:length(nv)
   n = nv(i);
   k = kv(i);
   if n >= 0
      if k >= 0
         if n >= k
            c = prod(1:n)/(prod(1:k)*prod(1:n-k));
         else
            c = 0;
        end
     else
        c = 0;
     end
   else
      if k >= 0
```

```
            c = (-1)^k * prod(1:k-n-1)/(prod(1:k)*prod(1:-n-1));
        else
            if n >= k
                c = (-1)^(n-k)*prod(1:-k-1)/(prod(1:n-k)*prod(1:-n-1));
            else
                c = 0;
            end
        end
    end
    a(i) = c;
end
```

dbleroots.m

```
function rts = dbleroots(p)

% Find roots of a polynomial with double roots:
% P(z) = Q(z)^2.
% Find roots of Q(z).
% Proceed by taking derivative of P(z) to improve
% the numerical accuracy of the root computation.

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N = length(p)-1;
p = p(:)';      % ensure p is a row vector
pdiff = p(1:N) .* (N:-1:1);

rts_p = roots(p);
rts_pdiff = roots(pdiff);

rts = rts_p;
for k = 1:N
    [tmp, i] = min(abs(rts(k)-rts_pdiff));
    rts(k) = rts_pdiff(i);
end

trim = zeros(N/2,1);
for i = 1:N/2
    trim(i) = rts(1);
    rts(1) = [];
    [tmp,k] = min(abs(trim(i)-rts));
    rts(k) = [];
end

rts = trim;
```

extractf.m

```
function f = extractf(h,p);
%
% f = extractf(h,p)
% find f such that h = conv(f,p)
%
% When such an f exists, this function has better
% numerical accuracy that the "deconv" command

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p = p(:);
h = h(:);
Np = length(p);
Nh = length(h);

C = convmtx(p,Nh-Np+1);

f = C\h;

% check accuracy of result:

SN = 0.000001;     % Small Number
e = max(abs(C*f - h));
% disp(e)
if e > SN
    disp('there is a problem in extracf')
    keyboard
end

f = f';
```

flip.m

```
function b = flip(a)

b = a(end:-1:1);
```

scalfn.m

```
function [s,t] = scalfn(h,J)
% [s,t] = scalfn(h,J);
% Scaling function obtained by dyadic expansion
% input
%    h : scaling filter
% output
%    s : samples of the scaling function phi(t)
%        for t = k/2^J, k=0,1,2,...
% % Example:
%    h = [1+sqrt(3) 3+sqrt(3) 3-sqrt(3) 1-sqrt(3)]/(4*sqrt(2));
%    [s,t] = scalfn(h);
%    plot(t,s)

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if nargin < 2
   J = 5;
end

N = length(h);
h = h(:).';               % form a row vector

% check sum rules
n  = 0:N-1;
e0 = sum(h) - sqrt(2);
e1 = sum(((-1).^n).*h);
if abs(e0) > 0.0001
   disp('   need: sum(h(n)) = sqrt(2)')
   return
end
if abs(e1) > 0.0001
   disp('   need: sum((-1)^n h(n)) = 0')
   return
end

% Make convolution matrix
H = toeplitz([h zeros(1,N-1)]',[h(1) zeros(1,N-1)]);
```

```
% or: H = convmtx(h(:),N);

% Make P matrix
P = sqrt(2)*H(1:2:2*N-1,:);

% Solve for vector
s = [P-eye(N); ones(1,N)] \ [zeros(N,1); 1];
s = s.';                        % phi at integers
L = N;                          % length of phi vector

% Loop through scales
for k = 0:J-1
  s = sqrt(2)*conv(h,s);
  L = 2*L-1;
  s = s(1:L);
  h = up(h,2);
end

% Time axis
t = (0:L-1)*(N-1)/(L-1);
```

up.m

```
function y = up(x,M)
% y = up(x,M)
% M-fold up-sampling of a 1-D signal

[r,c] = size(x);
if r > c
   y = zeros(M*r,1);
else
   y = zeros(1,M*c);
end
y(1:M:end) = x;
```

wletfn.m

```
function [w,s,t] = wletfn(h0,h1,K);

% [w,s,t] = wletfn(h0,h1,K);
%
% Computes the scaling function and wavelet
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if nargin < 3
    K = 7;
end

N0 = length(h0);
N1 = length(h1);

[s,t] = scalfn(h0,K);

L = length(s);

w = sqrt(2)*conv(up(h1,2^(K-1)),s(1:2:L));

L = (N0-1)/2 + (N1-1)/2;

t = [0:2^K*L]/2^K;

w = w(1:(2^K*L+1));
```

z2x.m

```
function p = z2x(h)
% p = z2x(h)
% Implements the change of variables
% x = (-z + 2 - 1/z)/4
% where h(z) is a odd-length symmetric filter

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N = length(h);
M = (N-1)/2;
p = [];
g = 1;
for k = 1:M
   g = conv(g,[-1 2 -1]/4);
end
for k = 0:M
   [q,r] = deconv(h,g);
   p(M+1-k) = q;
   h = r(2:end-1);
   g = deconv(g,[-1 2 -1]/4);
end
p = p(end:-1:1);
```