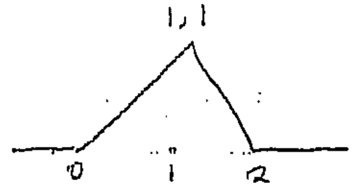
HW 8

① (a) Let h = pdf of $X_1 + X_2$. Then h = 

$g = h * f$, where f = pdf of X_3 $= 1$ on $(0,1)$
$= 0$, else

By class,

$$g(x) = \int_{x-1}^{x} h(y)\, dy, \quad -\infty < x < \infty$$

For $x < 0$, $g(x) = 0$

For $0 < x < 1$, $g(x) = \int_0^x y\, dy = \frac{1}{2} x^2$

For $1 < x < 2$, $g(x) = \int_{x-1}^{1} y\, dy + \int_1^x (2-y)\, dy$

So $g(x) = \frac{1}{2} y^2 \Big|_{x-1}^{1} - \frac{1}{2}(2-y)^2 \Big|_1^x$

$$= \frac{1}{2} - \frac{1}{2}(x-1)^2 + \frac{1}{2} - \frac{1}{2}(2-x)^2$$

$$= 1 - \frac{1}{2}(x^2 - 2x + 1) - \frac{1}{2}(4 - 4x + x^2)$$

$$= 3x - x^2 - 3/2, \quad 1 < x < 2$$

For $2 < x < 3$, $g(x) = \int_{x-1}^{2} (2-y)\, dy = -\frac{1}{2}(2-y)^2 \Big|_{y=x-1}^{y=2}$

$$= \frac{1}{2}(2-(x-1))^2$$

$$= \frac{1}{2}(3-x)^2$$

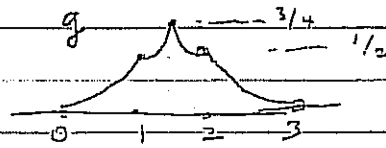
So: $g(x) = 0$, if $x < 0$

$= \frac{1}{2}x^2$, if $0 < x < 1$

$= 3x - x^2 - 3/2$, $1 < x < 2$

$= \frac{1}{2}(3-x)^2$, $2 < x < 3$

$= 0$, $x > 3$.

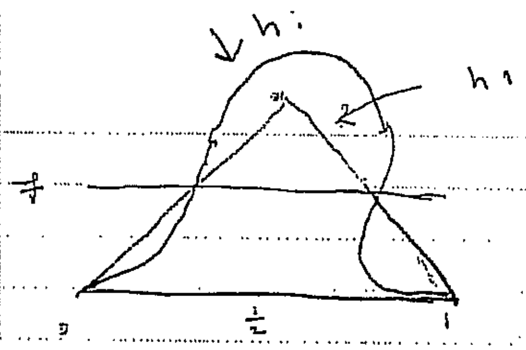


Note $3x - x^2 - \frac{3}{2} = \frac{9}{4} - \left(\frac{3}{2} - x\right)^2$

(b) $Z = \frac{1}{3}Y$. By class, $h(x) = 3g(3x)$.

Let $h_1(x) =$ pdf of $\frac{1}{2}(X_1 + X_2)$. Then $h_1(x) = 2f(2x)$

For graph of f, h_1 and h, see next page.



Rough sketch

Graphs become more and more peaked around

$x = \frac{1}{2}$

(2)

$x \backslash y$	1	2	
1	1/8	1/4	3/8
2	1/8	1/2	5/8
	1/4	3/4	

(a) $p_{X|Y}(1|1) = \dfrac{p(1,1)}{p_Y(1)} = \dfrac{1/8}{1/4} = \dfrac{1}{2}$

$p_{X|Y}(2|1) = \dfrac{p(2,1)}{p_Y(1)} = \dfrac{1/8}{1/4} = \dfrac{1}{2}$

$p_{X|Y}(1|2) = \dfrac{p(1,2)}{p_Y(2)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$

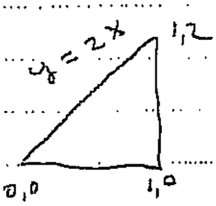
$p_{X|Y}(2|2) = \dfrac{p(2,2)}{p_Y(2)} = \dfrac{1/2}{3/4} = 2/3$

(b) Not indep. $p_X(1) = 3/8$, $p_{X|Y}(1|1) = \frac{1}{2}$

(c) $XY \le 3$ for all pts except $(2,2)$. $\boxed{P(XY \le 3) = \frac{1}{2}}$

$X+Y > 2$ occurs unless point $= (1,1)$. $\boxed{P(X+Y>2) = \frac{7}{8}}$

$X/Y > 1$ for $(2,1)$ only. $\boxed{P(X/Y > 1) = 1/8}$

(3) 

$f = \frac{5}{2}x^2 y$ in T

(a) cond. distn. exists $\Longleftrightarrow$ $0 < y < 2$

Fix $0 < y < 2$.

$f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

$$f_Y(y) = \int_{\frac{1}{2}y}^{1} \frac{5}{2} x^2 y \, dx = \frac{5}{2} y \left. \frac{x^3}{3} \right|_{x=\frac{1}{2}y}^{x=1}$$

$$= \frac{5}{6} y \left(1 - y^3/8\right)$$

$$f_{X|Y}(x|y) = \frac{5/2\, x^2 y}{5/6\, y(1 - y^3/8)} = \frac{3x^2}{1 - y^3/8}, \quad \text{if } \frac{y}{2} < x < 1$$

$$= 0, \quad \text{other } x$$

(b) $$f_X(x) = \int_0^{2x} \frac{5}{2} x^2 y \, dy$$

$$= \frac{5}{2} x^2 \left. \frac{y^2}{2} \right|_0^{2x} = \frac{5}{2} x^2 \frac{4x^2}{2} = 5x^4, \quad 0 < x < 1$$

cond distn exists $\Longleftrightarrow$ $0 < x < 1$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{5x^4} = \frac{5/2\, x^2 y}{5x^4} = \frac{1}{2}\, y/x^2, \quad 0 < y < 2x$$

$$= 0, \quad \text{other } y$$

(c) $$P\left(X < \tfrac{3}{4} \,\middle|\, Y = 1\right) = \int_0^{3/4} f_{X|Y}(x|1) \, dx$$

$$= \int_{1/2}^{3/4} \frac{3x^2}{7/8} \, dx = \left. \frac{8}{7} x^3 \right|_{1/2}^{3/4}$$

$$= \frac{8}{7}\left[\frac{27}{64} - \frac{1}{8}\right] = \frac{8}{7} \cdot \frac{19}{64} = \frac{19}{56}$$

$$\cong .339$$

$y^2 - 2y + 1$

4. Median is Y_3. Need to find d.f. of Y_3.

For $y \in (0,1)$, the event $Y_3 \le y$ occurs $\iff$

at least three of the X_i are $\le y$. So

$$P(Y_3 \le y) = P(\text{exactly three of } X_1, \ldots, X_5 \text{ are } \le y)$$
$$+ P(\text{exactly four}) + P(\text{exactly five})$$
$$= \binom{5}{3} y^3 (1-y)^2 + \binom{5}{4} y^4 (1-y) + y^5$$
$$10(y^5 - 2y^4 + y^3) + 5(y^4 - y^5) + y^5 = 6y^5 - 15y^4 + 10y^3$$

where we used $P(X_i \le y) = y$, for $y \in (0,1)$, $1 \le i \le 5$.

Then $P(\text{median} \in (a,b)) = P(Y_3 \le b) - P(Y_3 \le a)$

5. (a) $F(x) = x^2$, $0 \le x \le 1$, $\quad f(x) = 2x$, on $(0,1)$
$= 0$, $x \le 0$ $\quad = 0$, else
$= 1$, $x \ge 1$

(b) $P(\max\{X_1, X_2\} \le \frac{2}{3}) = \left[P(X_1 \le \frac{2}{3})\right]^2 = \left(\frac{4}{9}\right)^2 = .197$

(c) $P(\min\{X_1, X_2\} \le \frac{2}{3}) = 1 - P(\min\{X_1, X_2\} \ge \frac{2}{3})$
$$= 1 - \left[P(X_1 \ge \tfrac{2}{3})\right]^2 = 1 - \left(1 - \left(\tfrac{2}{3}\right)^2\right)^2$$
$$= 56/81 = .69$$

(d) $P(\min \ge \frac{1}{2}, \max \le \frac{2}{3}) = P\left(X_1 \text{ and } X_2 \text{ are both in } \left(\frac{1}{2}, \frac{2}{3}\right)\right)$
$$= P\left(X_1 \in \left(\tfrac{1}{2}, \tfrac{2}{3}\right)\right)^2 = \left[\left(\tfrac{2}{3}\right)^2 - \left(\tfrac{1}{2}\right)^2\right]^2 = \left(\tfrac{7}{36}\right)^2 = .0378$$