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Graduate Analysis

Midterm

November 4th

means $\mu(\mathbb{R}) < \infty$?

Question 1. *(60pts) Let μ and ν be two finite Borel measures on $\mathbb{R}$. For any Borel set $A \subset \mathbb{R}$, define*

$$\mu * \nu(A) = \mu \times \nu\left(\{(x,y) \in \mathbb{R}^2 : x + y \in A\}\right).$$

(a) Justify that $\mu * \nu$ is a finite Borel measure in $\mathbb{R}$ (Make a clear outline of everything that needs to be checked. You may take for granted that the set $\{(x,y) \in \mathbb{R}^2 : x+y \in A\}$ is Borel when A is Borel).

(b) Prove that if either μ or ν is absolutely continuous with respect to the Lebesgue measure, then $\mu * \nu$ is absolutely continuous respect to the Lebesgue measure as well.

(c) Assume that both μ and ν are absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivatives f and g respectively. Prove that $f * g$ is the Radon-Nikodym derivative of $\mu * \nu$.

(a) To show:

1. Finite. $\underbrace{\mu * \nu(\mathbb{R})}_{= (\mu\times\nu)(\mathbb{R}^2)} < \infty$

This is true since $(\mu \times \nu)(\mathbb{R}^2) = \mu(\mathbb{R})\,\nu(\mathbb{R}) < \infty$.

2. Borel: Borel sets are measurable.

Need to show countable ~~sub~~additivity. (??)

$\{A_i\}$ Borel $\Longrightarrow (\mu * \nu)(\bigcup A_i) \le \sum (\mu * \nu)(A_i)$

- First, $\mu \times \nu$ is defined on Borel sets of $\mathbb{R}^2$ by "you may take for granted..."

so $\mu * \nu$ is defined on Borel sets

- For countable subadd.

$$(\mu * \nu)\left(\bigcup_i A_i\right)$$

$$\|$$

$$(\mu\times\nu)\{(x,y) \mid x+y \in \bigcup A_i\}$$

$$= (\mu\times\nu)\left(\bigcup_i \{(x,y) \mid x+y\in A_i\}\right)$$

$$\leq \sum_i (\mu\times\nu)\left(\{(x,y) \mid x+y \in A_i\}\right)$$

countable subadditivity of $\mu\times\nu$.

$$= \sum_i (\mu * \nu)(A_i)$$

Is this enough? $(\mu * \nu(\emptyset) = 0$ is obvious$)$

$\mu\times\nu(\emptyset)$

NO, you proved it is an outer measure, not a measure.

(b) $\mu \ll \mathcal{L}$ means $\mu(A) = 0 \Leftarrow \mathcal{L}(A) = 0$.

s'pose $\mu \ll \mathcal{L}$. Now s'pose $\mathcal{L}(A) = 0$

Then $(\mu * \nu)(A) = (\mu\times\nu)\left(\{(x,y)\in\mathbb{R}^2 \mid x+y\in A\}\right)$

$$= \int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}} \mathbb{1}_{\{(x,y) \mid x+y\in A\}}\, d\mu(x)\, d\nu(y)$$

Fubini

$$= \int_{y\in\mathbb{R}} \mu\{x \mid x+y\in A\}\, d\nu(y)$$

$\mathcal{L}$ is transl inv so

draft

$$\mathcal{L}(\{x \mid x+y\in A\}) = \mathcal{L}(A-y) = \mathcal{L}(A) = 0$$

so by abs cty: $\mu(\{x \mid x+y\in A\}) = 0$

so $(\mu * \nu)(A) = 0$.

(c) $$\int \phi \, d\mu = \int \phi f \, d\mathcal{L}$$

$$\int \phi \, d\nu = \int \phi g \, d\mathcal{L}.$$

To show $\int \phi \, d(\mu * \nu) = \int \phi \, (f*g) \, d\mathcal{L}$.

How to do this???. Take $\phi = \mathbb{1}_A$. A meas'ble

$$\int \mathbb{1}_A \, d(\mu*\nu) = \mu * \nu \, (A)$$

$$= \int_{y\in\mathbb{R}} \mu(A-y) \, d\nu(y)$$

$$\int \mathbb{1}_A \, f*g \, d\mathcal{L} = \iint_{\{(x,y) \mid x+y\in A\}} f(x) g(y-x) \, dy \, dx$$

$$= \int_{y\in\mathbb{R}} \left(\int_{x\in A-y} f(x) \, d\mathcal{L}(x) \right) d\nu(y)$$

$$= \int_{y \in \mathbb{R}} \int_{x \in A-y} f(x) \, d\mathcal{L}(x) \quad d\nu(y)$$

$$= \int_{y \in \mathbb{R}} \int_{x \in A} f(x-y) \, d\mathcal{L}(x) \quad d\nu(y)$$ change of variables.

$$= \int_{x \in A} \int_{y \in \mathbb{R}} f(x-y) \, d\nu(y) \quad d\mathcal{L}(x)$$

RN derivative

$$= \int_{x \in A} \int_{y \in \mathbb{R}} f(x-y) \, g(y) \, d\mathcal{L}(y) \quad d\mathcal{L}(x)$$

$$= \int_{x \in A} (f * g)(x) \, d\mathcal{L}(x)$$

so we showed

$$\int_A d(\mu * \nu) = \int_A (f * g) \, d\mathcal{L}$$

I think that's enough... ???

I guess so.

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Question 2. *(40pts) Let $f_n : [0,1] \to \mathbb{R}$ be a bounded sequence of functions in $L^2([0,1])$. That means that there exists $C_0 > 0$ so that*

$$\|f_n(x)\|_{L^2} \le C_0 \quad \textit{for all } n.$$

Assume that f_n converges to f in $L^1([0,1])$. Prove that $f \in L^2([0,1])$ and $\|f\|_{L^2} \le C_0$.

$f_n \to f$ in L^1 $\quad \|f_n\|_2 \le C_0$

~~Maybe try Harnack's inequality.~~ $\|f_n - f\| \ldots$?

$$\int |f|^2 = \int |f - f_n + f_n|^2$$

$$= \int_0^1 |f-f_n|^2 + 2\int_0^1 |f_n||f-f_n| + \int_0^1 |f_n|^2 \ldots$$

$$\|f\|_{L^2} \le \|f-f_n\|_{L^2} + \|f_n\|_{L^2}$$

start here

Is it true that $f_n \xrightarrow{L^2} f$??

or maybe for some subsequence?

You are using heavier artillery than what you are supposed to.

Wait, yes! $\quad \sup \|f_n\|_{L^2} < \infty$

$\underset{\text{Banach Alaoglu}}{\Longrightarrow}$ $\exists$ subseq f_{n_i} s.t.

$f_{n_i} \rightharpoonup g$ in L^2 for some $g \in L^2$

Then $\forall \phi \in L^\infty (\subset L^2 = (L^2)^*)$ we have

since $[0,1]$ bounded

$$\int_0^1 \phi f_{n_i} \to \int_0^1 \phi g.$$

uniform boundedness principle $\Longrightarrow \|g\|_{L^2} \le C_0$

since $L^\infty = (L^1)^*$, it follows that $f_{n_i} \rightharpoonup g$ in L^1

But
$$\left.\begin{aligned} f_{n_i} &\longrightarrow g \quad \text{in } L^1 \\ f_n &\longrightarrow f \quad \text{in } L^1 \end{aligned}\right\} \Longrightarrow f=g.$$

So we've shown $f \in L^2$ and $\|f\|_{L^2} \leq C_0$,

I think...

(I am not as good as drawing as some people in your class... !! $\frown$)