

EXTRA EXAM

Math 217

Final exam

December 12, 2008

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Course: *Math 217*

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TABLE 9.3.1 Stability and Instability Properties of Linear and Almost Linear Systems

r_1, r_2	Linear System		Almost Linear System	
	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

Even if the critical point is of the same type as that of the linear system, the trajectories of the almost linear system may be considerably different in appearance from those of the corresponding linear system, except very near the critical point. However, it can be shown that the slopes at which trajectories “enter” or “leave” the critical point are given correctly by the linear equation.

Damped Pendulum. We continue the discussion of the damped pendulum begun in Examples 2 and 3. Near the origin the nonlinear equations (8) are approximated by the linear system (16), whose eigenvalues are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}. \quad (18)$$

The nature of the solutions of Eqs. (8) and (16) depends on the sign of $\gamma^2 - 4\omega^2$ as follows:

1. If $\gamma^2 - 4\omega^2 > 0$, then the eigenvalues are real, unequal, and negative. The critical point $(0, 0)$ is an asymptotically stable node of the linear system (16) and of the almost linear system (8).
2. If $\gamma^2 - 4\omega^2 = 0$, then the eigenvalues are real, equal, and negative. The critical point $(0, 0)$ is an asymptotically stable (proper or improper) node of the linear system (16). It may be either an asymptotically stable node or spiral point of the almost linear system (8).
3. If $\gamma^2 - 4\omega^2 < 0$, then the eigenvalues are complex with negative real part. The critical point $(0, 0)$ is an asymptotically stable spiral point of the linear system (16) and of the almost linear system (8).

1. Which of the following matrices have real eigenvalues?

I) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ II) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ III) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

A) I

B) II

C) III

D) I, II

E) I, III

F) II, III

G) I, II, III

H) None of them

I) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is diagonal so $r_1=1$ $r_2=2$.

II) $r^2 - r + (-1) = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

III) $r^2 - 0r + 1 = 0$

$$r = \pm i$$

2. Let $w(t) = W\left(\begin{pmatrix} t \\ 1 \end{pmatrix}, \begin{pmatrix} t^2 \\ 2t \end{pmatrix}\right)$, the Wronskian of the vector functions $\begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\begin{pmatrix} t^2 \\ 2t \end{pmatrix}$. Find $w(2)$.

A) 0

B) 1/3

C) 1/2

D) 1

E) 3/2

F) 2

G) 5/2

H) 3

I) 4

J) None of the above

$$w(t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$$

$$w(2) = 2^2 = 4$$

3. Which of the following vector functions $\vec{x}(t)$ are solutions of $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}$?

I) $\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$ II) $\begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix}$ III) $\begin{pmatrix} e^t \\ 3e^t \end{pmatrix}$

A) I

$$r^2 + 4r + 3 = 0$$

B) II

$$(r+3)(r+1) = 0$$

C) III

$$r = -1, -3$$

D) I, II

E) I, III

$$\begin{array}{l} r_1 = -1 \\ (A - (-1)I) \vec{\xi}_1 = 0 \end{array}$$

$$\begin{array}{l} r_2 = -3 \\ (A - (-3)I) \vec{\xi}_2 = 0 \end{array}$$

F) II, III

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

G) I, II, III

$$-x_1 + x_2 = 0$$

$$x_1 + x_2 = 0$$

H) None of them

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{\xi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

$$= c_1 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix}$$

4. Solve the I. V. P. $\bar{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \bar{x}$, $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

A) $\begin{pmatrix} 3 - 2e^{-2t} \\ 4 - 4e^{-2t} \end{pmatrix}$

$$r^2 + 2r = 0$$

$$r(r+2) = 0$$

B) $\begin{pmatrix} 2 + 3e^{-t} \\ 1 - 2e^{-t} \end{pmatrix}$

$$r = 0, -2$$

C) $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$\underline{r_1 = 0}$$

$$(A - 0I) \vec{v}_1 = 0$$

$$\underline{r_2 = -2}$$

$$(A - (-2)I) \vec{v}_2 = 0$$

D) $\begin{pmatrix} 1 + 4t \\ 8t \end{pmatrix}$

$$\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

E) $\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

$$4x_1 - 3x_2 = 0$$

$$6x_1 - 3x_2 = 0$$

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$2x_1 - x_2 = 0$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

F) $\begin{pmatrix} \frac{1}{4}e^t \\ -2e^t \end{pmatrix}$

G) $\begin{pmatrix} t \\ t+1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

H) $\begin{pmatrix} \cos 2t \\ 3 \sin 2t \end{pmatrix}$

$$c_1 = 1 \quad c_2 = -2$$

I) $\begin{pmatrix} -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

$$\vec{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + -2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$$

$$= \begin{pmatrix} 3 - 2e^{-2t} \\ 4 - 4e^{-2t} \end{pmatrix}$$

J) None of the above

5. Which of the following are solutions of $\vec{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$?

I) $\begin{pmatrix} 2e^{-t} \cos 2t \\ e^{-t} \sin 2t \end{pmatrix}$ II) $\begin{pmatrix} -2e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}$ III) $\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$

A) I

B) II

C) III

D) I, II

E) I, III

F) II, III

G) I, II, III

H) None of them

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

$$\lambda = -1 \quad \mu = 2$$

$$(A - (-1 + 2i)I) \vec{\phi} = 0$$

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_1 - 2ix_2 = 0$$

$$\vec{\phi} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i$$

$$\vec{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{x} = c_1 e^{-t} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right) + c_2 e^{-t} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right)$$

6. Solve the I. V. P. $\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

A) $\begin{pmatrix} 3 - 2e^{-2t} \\ 4 - 4e^{-2t} \end{pmatrix}$

$$r^2 + 1 = 0$$

$$r = \pm i$$

B) $\begin{pmatrix} 2 + 3e^{-t} \\ 1 - 2e^{-t} \end{pmatrix}$

$$\lambda = 0 \quad \mu = 1$$

C) $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$(A - iI) \vec{\psi} = 0$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

D) $\begin{pmatrix} 1 + 4t \\ 8t \end{pmatrix}$

$$x_1 - i x_2 = 0$$

$$\vec{\psi} = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$$

E) $\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

$$\vec{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

F) $\begin{pmatrix} \frac{1}{4} e^t \\ -2e^t \end{pmatrix}$

$$\vec{x} = c_1 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right) + c_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t \right)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$c_1 = 0 \quad c_2 = 1$$

G) $\begin{pmatrix} t \\ t+1 \end{pmatrix}$

$$\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

H) $\begin{pmatrix} \cos 2t \\ 3 \sin 2t \end{pmatrix}$

I) $\begin{pmatrix} -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

J) None of the above

7. Solve the I. V. P. $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

A) $\begin{pmatrix} 3 - 2e^{-2t} \\ 4 - 4e^{-2t} \end{pmatrix}$

$$r^2 = 0$$

B) $\begin{pmatrix} 2 + 3e^{-t} \\ 1 - 2e^{-t} \end{pmatrix}$

$$(A - 0I) \vec{\xi} = 0$$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

C) $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$4x_1 - 2x_2 = 0$$

$$2x_1 - x_2 = 0$$

D) $\begin{pmatrix} 1 + 4t \\ 8t \end{pmatrix}$

$$\vec{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

E) $\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

$$(A - 0I) \vec{\eta} = \vec{\xi}$$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

F) $\begin{pmatrix} \frac{1}{4}e^t \\ -2e^t \end{pmatrix}$

$$4\eta_1 - 2\eta_2 = 1$$

$$\vec{\eta} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

G) $\begin{pmatrix} t \\ t+1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right)$$

H) $\begin{pmatrix} \cos 2t \\ 3 \sin 2t \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$c_1 = 1 \quad c_2 = 4$$

I) $\begin{pmatrix} -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right)$$

J) None of the above

$$= \begin{pmatrix} 1 + 4t \\ 8t \end{pmatrix}$$

8. Solve the I. V. P. $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

A) $\begin{pmatrix} 3 - 2e^{-2t} \\ 4 - 4e^{-2t} \end{pmatrix}$

$$x' = 2x$$

$$x(0) = 1$$

$$y' = 2y$$

$$y(0) = 1$$

B) $\begin{pmatrix} 2 + 3e^{-t} \\ 1 - 2e^{-t} \end{pmatrix}$

$$x = e^{2t}$$

$$y = e^{2t}$$

C) $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

D) $\begin{pmatrix} 1 + 4t \\ 8t \end{pmatrix}$

E) $\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

F) $\begin{pmatrix} \frac{1}{4}e^t \\ -2e^t \end{pmatrix}$

G) $\begin{pmatrix} t \\ t+1 \end{pmatrix}$

H) $\begin{pmatrix} \cos 2t \\ 3 \sin 2t \end{pmatrix}$

I) $\begin{pmatrix} -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

J) None of the above

9. The general solution of $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$ is $\vec{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$. Use the method of undetermined coefficients to find the general solution of the nonhomogeneous equation $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$. $\vec{x} = \vec{x}^{(c)} + ?$:

A) $\begin{pmatrix} 3 - 2e^{-2t} \\ 4 - 4e^{-2t} \end{pmatrix}$

$$\vec{x}_p = \vec{a} e^t$$

B) $\begin{pmatrix} 2 + 3e^{-t} \\ 1 - 2e^{-t} \end{pmatrix}$

$$\vec{x}_p' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}_p + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$

$$\vec{a} e^t = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{a} e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$

C) $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$\vec{a} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{a} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \vec{a}$$

D) $\begin{pmatrix} 1 + 4t \\ 8t \end{pmatrix}$

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

E) $\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

$$-2 = a_2$$

$$-1 = 4a_1$$

$$\vec{a} = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix}$$

F) $\begin{pmatrix} \frac{1}{4} e^t \\ 4 \\ -2e^t \end{pmatrix}$

$$\vec{x}_p = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix} e^t = \begin{pmatrix} \frac{1}{4} e^t \\ -2e^t \end{pmatrix}$$

G) $\begin{pmatrix} t \\ t+1 \end{pmatrix}$

H) $\begin{pmatrix} \cos 2t \\ 3 \sin 2t \end{pmatrix}$

I) $\begin{pmatrix} -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

J) None of the above

10. Solve the I. V. P. $\bar{x}' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\bar{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (There is no misprint here.)

A) $\begin{pmatrix} 3 - 2e^{-2t} \\ 4 - 4e^{-2t} \end{pmatrix}$

$$\begin{array}{ll} x' = 1 & x(0) = 0 \\ y' = 1 & y(0) = 1 \end{array}$$

B) $\begin{pmatrix} 2 + 3e^{-t} \\ 1 - 2e^{-t} \end{pmatrix}$

$$\begin{array}{l} x = t \\ y = t + 1 \end{array}$$

C) $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

D) $\begin{pmatrix} 1 + 4t \\ 8t \end{pmatrix}$

E) $\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$

F) $\begin{pmatrix} \frac{1}{4}e^t \\ -2e^t \end{pmatrix}$

G) $\begin{pmatrix} t \\ t + 1 \end{pmatrix}$

H) $\begin{pmatrix} \cos 2t \\ 3 \sin 2t \end{pmatrix}$

I) $\begin{pmatrix} -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

J) None of the above

11. Classify the critical point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the system $\vec{x}' = \begin{pmatrix} -4 & -4 \\ 0 & -1 \end{pmatrix} \vec{x}$.

A) (Stable) center

B) (Unstable) saddle point

C) Asymptotically stable improper node

D) Unstable improper node

E) Asymptotically stable spiral point

F) Unstable spiral point

G) Unstable proper node

H) Either a node or spiral point, which is unstable

I) Either a node or spiral point, which is asymptotically stable

J) None of the above

A is upper triangular, so

$$r_1 = -4 \quad r_2 = -1$$

$$r_1 < r_2 < 0$$

12. Which of the following are critical points of $\begin{cases} dx/dt = -x + 2xy \\ dy/dt = y - x^2 - y^2 \end{cases}$?

I) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ II) $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ III) $\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$

A) I

B) II

C) III

D) I, II

E) I, III

F) II, III

G) I, II, III

H) None of them

$$\begin{cases} -x + 2xy = 0 \\ y - x^2 - y^2 = 0 \end{cases}$$

$$-x + 2xy = 0$$

$$x(-1 + 2y) = 0$$

Case I $x = 0$

$$y - y^2 = 0$$

$$y = 0, 1$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Case 2 $-1 + 2y = 0$

$$y = \frac{1}{2}$$

$$\frac{1}{2} - x^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$\frac{1}{4} = x^2$$

$$\pm \frac{1}{2} = x$$

$$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$

13. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point of the almost linear system $\begin{cases} \frac{dx}{dt} = x + y^2 \\ \frac{dy}{dt} = x + y \end{cases}$. Classify it.

A) (Stable) center

$$F = x + y^2$$

B) (Unstable) saddle point

$$G = x + y$$

C) Asymptotically stable improper node

$$F_x = 1 \quad F_y = 2y$$

D) Unstable improper node

$$G_x = 1 \quad G_y = 1$$

E) Asymptotically stable spiral point

F) Unstable spiral point

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

G) Unstable proper node

$$r^2 - 2r + 1 = 0$$

H) Either a node or spiral point, which is unstable

$$(r-1)^2 = 0$$

I) Either a node or spiral point, which is asymptotically stable

$$r_1 = r_2 = 1 > 0$$

J) None of the above

14. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a critical point of the almost linear system $\begin{cases} dx/dt = 1-y \\ dy/dt = x^2 - y^2 \end{cases}$. Classify it.

A) (Stable) center

B) (Unstable) saddle point

C) Asymptotically stable improper node

D) Unstable improper node

E) Asymptotically stable spiral point

F) Unstable spiral point

G) Unstable proper node

H) Either a node or spiral point, which is unstable

I) Either a node or spiral point, which is asymptotically stable

J) None of the above

$$F = 1 - y$$

$$G = x^2 - y^2$$

$$F_x = 0 \quad F_y = -1$$

$$G_x = 2x \quad G_y = -2y$$

$$\begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix}$$

$$r^2 + 2r - 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4+8}}{2}$$

$$= \frac{-2 \pm \sqrt{12}}{2}$$

$$\sqrt{12} > \sqrt{4} = 2$$

So $r_1 < 0 < r_2$

15. The competing species equations $\begin{cases} x' = (1-x-y)x \\ y' = \left(\frac{3}{4} - \frac{1}{2}x - y\right)y \end{cases}$ have a critical point (and hence a possible coexistence point at $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$). Classify it.

A) (Stable) center

B) (Unstable) saddle point

C) Asymptotically stable improper node

D) Unstable improper node

E) Asymptotically stable spiral point

F) Unstable spiral point

G) Unstable proper node

H) Either a node or spiral point, which is unstable

I) Either a node or spiral point, which is asymptotically stable

J) None of the above

$$F = (1-x-y)x$$

$$G = \left(\frac{3}{4} - \frac{1}{2}x - y\right)y$$

$$F_x = -x + (1-x-y) \quad F_y = -x$$

$$G_x = -\frac{1}{2}y \quad G_y = -y + \left(\frac{3}{4} - \frac{1}{2}x - y\right)$$

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

$$r^2 + r + \frac{1}{8} = 0$$

$$r = \frac{-1 \pm \sqrt{1 - \frac{1}{2}}}{2} = \frac{-1 \pm \sqrt{\frac{1}{2}}}{2}$$

$$\sqrt{\frac{1}{2}} < \sqrt{1} = 1 \text{ so } r_1 < r_2 < 0$$

16. The predator – prey (Lotka – Volterra) equations $\begin{cases} x' = (4-2y)x \\ y' = (-3+x)y \end{cases}$ have critical points only at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Classify the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

A) (Stable) center

B) (Unstable) saddle point

C) Asymptotically stable improper node

D) Unstable improper node

E) Asymptotically stable spiral point

F) Unstable spiral point

G) Unstable proper node

H) Either a node or spiral point, which is unstable

I) Either a node or spiral point, which is asymptotically stable

J) None of the above

$$F = (4-2y)x$$

$$G = (-3+x)y$$

$$F_x = (4-2y)$$

$$F_y = -2x$$

$$G_x = y$$

$$G_y = (-3+x)$$

$$\begin{pmatrix} 0 & -6 \\ 2 & 0 \end{pmatrix}$$

$$r^2 + 12 = 0$$

$$r = \pm 2\sqrt{3}i$$

So center or spiral point - but we

know predator prey is always a center

17. Find the Fourier coefficient b_1 of $f(x) = \sin x$.

A) 0

B) $1/3$

C) $1/2$

D) 1

E) $3/2$

F) 2

G) $5/2$

H) 3

I) 4

J) None of the above

$$\begin{aligned} b_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin x dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin^2 x dx \\ &= \frac{2}{\pi} \int_0^{\pi} \left(\frac{1 - \cos 2x}{2} \right) dx \\ &= \frac{1}{\pi} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi} \\ &= \frac{1}{\pi} \left((\pi - \frac{1}{2} \cdot 0) - (0 - \frac{1}{2} \cdot 0) \right) \\ &= 1 \end{aligned}$$

18. Let $f(x) = x$ on $[-\pi, \pi]$, extended periodically to all of \mathbb{R} . Find the Fourier coefficient b_1 of $f(x)$.

A) 0

B) $1/3$

C) $1/2$

D) 1

E) $3/2$

F) 2

G) $5/2$

H) 3

I) 4

J) None of the above

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx$$

$$\begin{array}{ll} u = x & v = -\cos x \\ du = dx & dv = \sin x \, dx \end{array}$$

$$= \frac{2}{\pi} \left(-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \right)$$

$$= \frac{2}{\pi} \left(-x \cos x + \sin x \right) \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \left((-\pi \cdot -1 + 0) - (-0 + 0) \right)$$

$$= 2$$

19. Let $f(x)$ be the square wave function on $[-\pi, \pi]$, $f(x) = \begin{cases} -\pi & \text{if } -\pi \leq x < 0 \\ \pi & \text{if } 0 \leq x < \pi \end{cases}$, extended periodically to all of \mathbf{R} . To what value does the Fourier series of $f(x)$ converge when $x = 0$?

A) 0

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \pi = \pi$$

B) 1/3

C) 1/2

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -\pi = -\pi$$

D) 1

E) 3/2

$$\frac{\pi + (-\pi)}{2} = 0$$

F) 2

G) 5/2

H) 3

I) Fails to converge at $x = 0$ because $f(x)$ is not continuous there.

J) None of the above

20. Let $f(x) = x^3$ on $[-\pi, \pi]$, extended periodically to all of \mathbf{R} . Note that $f(x)$ is odd.
Find the Fourier coefficient a_2 for $f(x)$.

A) 0

B) $1/3$

C) $1/2$

D) 1

E) $3/2$

F) 2

G) $5/2$

H) 3

I) None of the above