Exam 1

Math 217

This exam consists of 16 questions worth 5 points each. You must show all work. Answers without work will receive no credit. Implicit solutions are acceptable where applicable.

For numbers 1-3, Consider the following four first order differential equations. One is separable, one is linear, one is exact, and one we have not discussed how to solve. Solve the three solvable ones.

$$\frac{dy}{dx} = 1 - \frac{y}{2x} \qquad \frac{dy}{dx} = -\frac{y^2}{2x}$$

$$\frac{dy}{dx} = \frac{1}{y} - \frac{y}{2x} \qquad \frac{dy}{dx} = \frac{1}{y} - \frac{y^2}{2x}$$

1. Solve the separable one.

$$\frac{dy}{dx} = \frac{-y^2}{2x}$$

$$\int \frac{-dy}{y^2} = \int \frac{dx}{2x}$$

$$\frac{1}{y} = \frac{1}{2}\ln|x| + C$$

$$\boxed{y = \frac{1}{\frac{1}{2}\ln|x| + C}}$$

2. Solve the linear one.

$$y' + \frac{1}{2x} y = 1$$

$$\mu = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln|x|} = x^{1/2}$$

$$x^{1/2}y' + \frac{1}{2}x^{-1/2}y = x^{1/2}$$

$$(x^{1/2}y)' = x^{1/2}$$

$$x^{1/2}y = \frac{2}{3}x^{3/2} + C$$

$$\boxed{y = \frac{2}{3}x + Cx^{-1/2}}$$

3. Solve the exact one.

$$\frac{dy}{dx} = \frac{1}{y} - \frac{y}{2x}$$

$$2xy\,dy = (2x - y^2)dx$$

$$(y^2 - 2x)dx + 2xy\,dy = 0$$

$$M_y = 2y$$

$$N_x = 2y$$

$$\psi = \int M\,dx + h(y)$$

$$= \int (y^2 - 2x)dx + h(y)$$

$$= xy^2 - x^2 + h(y)$$

$$\psi_y = N$$

$$2xy + h'(y) = 2xy$$

$$h'(y) = 0$$

$$h(y) = 0$$

$$\psi = xy^2 - x^2$$

$$\boxed{xy^2 - x^2 = C}$$

4. Use Euler's Method with $h = 0.2$ to estimate $y(3)$ in the following IVP:

$$y' + y = 3, y(2.4) = 1$$

$$y' = 3 - y$$

x	y	y'	Δy
2.4	1	2	0.4
2.6	1.4	1.6	0.32
2.8	1.72	1.28	0.256
3.0	1.976		

$$\boxed{y(3) \approx 1.976}$$

5. Compute the Wronskian $W(t^2e^t, te^t)$.

$$W(t^2e^t, te^t) = \begin{vmatrix} t^2e^t & te^t \\ 2te^t + t^2e^t & e^t + te^t \end{vmatrix} = \left(t^2e^{2t} + t^3e^{2t}\right) - \left(2t^2e^{2t} + t^3e^{2t}\right)$$

$$= \boxed{-t^2e^{2t}}$$

6. A 200L tank is full of a 20 g/L salt solution. Pure water is pumped in at 10 L/min, and the tank is drained at 10L/min. Find an expression for the amount of salt in the tank as a function of time.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} \qquad Q(0) = (200)(20) = 4000$$

$$= 0 - \frac{Q}{200} \cdot 10$$

$$= -\frac{Q}{20}$$

$$\frac{dQ}{Q} = -\frac{1}{20}\,dt$$

$$\ln|Q| = -\frac{1}{20}t + C$$

$$Q = Ae^{-\frac{1}{20}t}$$

$$4000 = Q(0) = A$$

$$\boxed{Q = 4000e^{-\frac{1}{20}t}}$$

7. Solve $(3x^2e^y + 2x(\cos y + 1))dx + (x^3e^y - x^2\sin y + 2e^y\cos y)dy = 0$

$M_y = 3x^2e^y - 2x\sin y$

$N_x = 3x^2e^y - 2x\sin y$

$\Psi = \int M\,dx + h(y) = \int 3x^2e^y + 2x(\cos y + 1)\,dx + h(y)$

$= x^3e^y + x^2(\cos y + 1) + h(y)$

$\Psi_y = N$

$x^3e^y - x^2\sin y + h'(y) = x^3e^y - x^2\sin y + 2e^y\cos y$

$h'(y) = 2e^y\cos y$

$h(y) = e^y\sin y + e^y\cos y$

$\Psi = x^3e^y + x^2(\cos y + 1) + e^y\sin y + e^y\cos y$

$$x^3e^y + x^2(\cos y + 1) + e^y\sin y + e^y\cos y = C$$

$\int e^y\cos y =$

$u = e^y \quad v = \sin y$

$du = e^y dy \quad dv = \cos y\,dy$

$= e^y\sin y - \int e^y\sin y\,dy$

$u = e^y \quad v = \cos y$

$du = e^y dy \quad dv = -\sin y\,dy$

$= e^y\sin y + e^y\cos y - \int e^y\cos y\,dy$

$2\int e^y\cos y = e^y\sin y + e^y\cos y$

8. Solve $(x^2y + 3xy + 2y + x^2 + 3x + 2)dx + (x + 2)dy = 0$.

$[y(x^2 + 3x + 2) + (x^2 + 3x + 2)]dx + (x+2)dy = 0$

$(y+1)(x+2)(x+1)\,dx + (x+2)\,dy = 0$

$(y+1)(x+1)\,dx + dy = 0$

$dy = -(y+1)(x+1)dx$

$\int \frac{dy}{y+1} = \int -(x+1)dx$

$\ln|y+1| = -\frac{1}{2}x^2 - x + C$

$y + 1 = Ae^{-\frac{1}{2}x^2 - x}$

$$y = -1 + Ae^{-\frac{1}{2}x^2 - x}$$

9. Find and classify all the equillibria of the following:

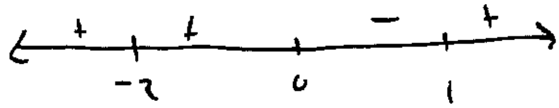
1	1	3	0	-4
		1	4	4
	1	4	4	0

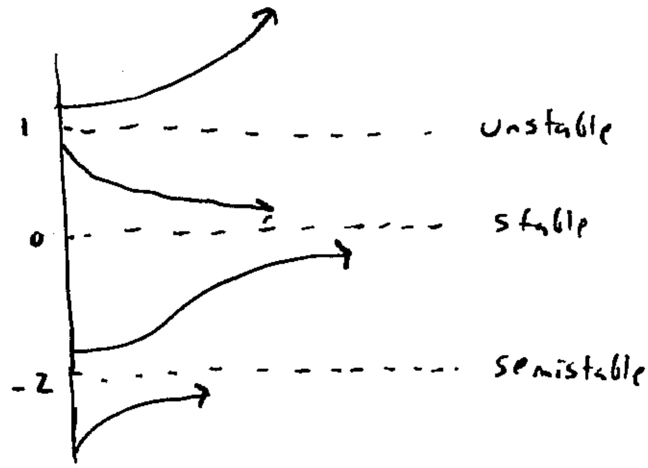
$$y' = y^6 + 3y^5 - 4y^3$$

$$= y^3(y^3 + 3y^2 - 4)$$

$$= y^3(y-1)(y^2 + 4y + 4)$$

$$= y^3(y-1)(y+2)^2$$





$y=1$	unstable
$y=0$	stable
$y=-2$	semi stable

10. Solve $y' = \frac{1}{9}(y+2)^2(y-1)$

$$\frac{9\,dy}{(y+2)^2(y-1)} = dt$$

$$\frac{9}{(y+2)^2(y-1)} = \frac{A}{(y+2)^2} + \frac{B}{y+2} + \frac{C}{y-1}$$

$$9 = A(y-1) + B(y+2)(y-1) + C(y+2)^2$$

$y=1 \Rightarrow 9 = 9C$
$1 = C$

$y=-2 \Rightarrow 9 = -3A$
$-3 = A$

coefficient of $y^2 \Rightarrow B + C = 0$
$B + 1 = 0$
$B = -1$

$$\int \left(\frac{-3}{(y+2)^2} - \frac{1}{y+2} + \frac{1}{y-1} \right) dy = \int dt$$

$$\frac{3}{y+2} - \ln|y+2| + \ln|y-1| = t + C$$

$$\boxed{\frac{3}{y+2} + \ln\left|\frac{y-1}{y+2}\right| = t + C}$$

11. A 2 kg mass is suspended from a spring, upon which the spring extends $\frac{10}{13}$ m. Friction in the spring exerts 8 N of force when the mass is traveling 1 m/s. The mass is pushed up $\frac{1}{2}$ m and released from rest. Give an expression for the motion of the mass as a function of time.

$K \cdot \frac{10}{13} = 2 \cdot g = 2 \cdot 10 = 20$

$K = \frac{13}{10} \cdot 20 = 26$

$8 = b \cdot 1$

$8 = b$

$mx'' + bx' + Kx = 0$

$2x'' + 8x' + 26x = 0$

$x'' + 4x' + 13x = 0$

$r^2 + 4r + 13 = 0$

$r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$

$x = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$

$x' = -2e^{-2t}(c_1 \cos 3t + c_2 \sin 3t) + 3e^{-2t}(-c_1 \sin 3t + c_2 \cos 3t)$

$\frac{1}{2} = x(0) = c_1$

$0 = x'(0) = -2c_1 + 3c_2$ $\quad c_2 = \frac{1}{3}$

$\boxed{x = e^{-2t}\left(\frac{1}{2}\cos 3t + \frac{1}{3}\sin 3t\right)}$

12. A 4 kg mass is suspended from a spring, upon which the spring extends 0.4 m. The system undergoes an external force of $F(t) = 36\cos(4t)$ and has no friction. If the mass is pulled down 0.5m and given an initial velocity of 5 m/s downwards, give an expression for the motion of the mass as a function of time.

$K \cdot 0.4 = 4 \cdot g = 4 \cdot 10 = 40$

$K = 100$

$mx'' + Kx = F(t)$

$4x'' + 100x = 36\cos(4t)$

$L(x) = x'' + 25x = 9\cos(4t)$

$r^2 + 25 = 0$

$x = c_1 \cos 5t + c_2 \sin 5t + x_p$

$x_p = A\cos 4t + B\sin 4t$

$x_p'' = -16A\cos 4t - 16B\sin 4t$

$L(x_p) = (-16A\cos 4t - 16B\sin 4t) + 25A\cos 4t + 25B\sin 4t = 9\cos 4t$

$9(A\cos 4t + B\sin 4t) = 9\cos 4t$

$A = 1 \quad B = 0$

$x_p = \cos 4t$

$x = c_1 \cos 5t + c_2 \sin 5t + \cos 4t$

$x' = -5c_1 \sin 5t + 5c_2 \cos 5t - 4\sin 4t$

$\frac{1}{2} = x(0) = c_1 + 1$

$5 = x'(0) = 5c_2$

$c_1 = -\frac{1}{2} \quad c_2 = 1$

$\boxed{x = -\frac{1}{2}\cos 5t + \sin 5t + \cos 4t}$

13. Solve $y'' - 6y' + 10y = \frac{e^{3t}}{\tan t}$

$r^2 - 6r + 10 = 0$

$r = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i$

$y = c_1 \underbrace{e^{3t}\cos t}_{y_1} + c_2 \underbrace{e^{3t}\sin t}_{y_2} + y_p$

$W(e^{3t}\cos t, e^{3t}\sin t) = \begin{vmatrix} e^{3t}\cos t & e^{3t}\sin t \\ -e^{3t}\sin t + 3e^{3t}\cos t & e^{3t}\cos t + 3e^{3t}\sin t \end{vmatrix}$

$= e^{6t}\left[(\cos^2 t + 3\sin t\cos t) - (-\sin^2 t + 3\sin t\cos t)\right]$

$= e^{6t}$

$\int \frac{y_2 g}{W} dt = \int \frac{(e^{3t}\sin t)\cdot \frac{e^{3t}}{\tan t}}{e^{6t}} dt = \int \frac{\sin t}{\tan t} dt = \int \cos t \, dt = \sin t$

$\int \frac{y_1 g}{W} dt = \int \frac{(e^{3t}\cos t)\cdot\frac{e^{3t}}{\tan t}}{e^{6t}} dt = \int \frac{\cos^2 t}{\sin t} dt = \int \frac{1-\sin^2 t}{\sin t} dt = \int \csc t - \sin t \, dt$

$= -\ln|\csc t + \cot t| + \cos t$

$y_p = -y_1 \int \frac{y_2 g}{W} dt + y_2 \int \frac{y_1 g}{W} dt = -e^{3t}\cos t(\sin t) + e^{3t}\sin t(-\ln|\csc t + \cot t| + \cos t)$

$= -e^{3t}\sin t \ln|\csc t + \cot t|$

$$\boxed{y = c_1 e^{3t}\cos t + c_2 e^{3t}\sin t - e^{3t}\sin t \ln|\csc t + \cot t|}$$

14. Solve $y'' - 4y = 2e^t + e^{2t}$

$r^2 - 4 = 0$

$r = \pm 2$

$y = c_1 e^{2t} + c_2 e^{-2t} + y_p$

$y_p = Ae^t + Bte^{2t}$

$y_p' = Ae^t + B(2te^{2t} + e^{2t})$

$y_p'' = Ae^t + B(4e^{2t} + 4te^{2t})$

$L(y_p) = [Ae^t + B(4e^{2t} + 4te^{2t})] - 4[Ae^t + Bte^{2t}] = 2e^t + e^{2t}$

$-3Ae^t + 4Be^{2t} = 2e^t + e^{2t}$

$A = -\frac{2}{3} \quad B = \frac{1}{4}$

$$\boxed{y = c_1 e^{2t} + c_2 e^{-2t} - \frac{2}{3}e^t + \frac{1}{4}te^{2t}}$$

15. Given that $y_1 = x$ is a solution, find the general solution of $(1-x)y'' + xy' - y = 0$. HINT: $\frac{d}{dx}\left(\frac{e^x}{x}\right) = e^x\left(\frac{x-1}{x^2}\right)$

$y = v \cdot x$

$y' = v'x + v$

$y'' = v''x + 2v'$

$(1-x)(v''x + 2v') + x(v'x + v) - (vx) = 0$

$x(1-x)v'' + (x^2 - 2x + 2)v' = 0$

Let $u = v'$

$(x - x^2)u' + (x^2 - 2x + 2)u = 0$

$\frac{du}{u} = \frac{x^2 - 2x + 2}{x^2 - x} dx = 1 + \frac{2-x}{x^2-x} dx$

$\frac{2-x}{x^2-x} = \frac{A}{x} + \frac{B}{x-1}$

$2 - x = A(x-1) + Bx$

$A = -2 \quad B = 1$

$\frac{du}{u} = \left(1 + \frac{1}{x-1} - \frac{2}{x}\right)dx$

$\ln|u| = x + \ln|x-1| - 2\ln|x| + C$

$u = Ae^{x \ln\left|\frac{x-1}{x^2}\right|}$

$= Ae^x\left(\frac{x-1}{x^2}\right)$

$v = \int u\,dx = A\int e^x\left(\frac{x-1}{x^2}\right)dx$

$= A \cdot \frac{e^x}{x} + B$

$= C_1 \cdot \frac{e^x}{x} + C_2$

$y = v \cdot x = \boxed{C_1 \cdot e^x + C_2 x}$

16. Given that $y_1 = e^x$ and $y_2 = x$ are solutions of the corresponding homogeneous equation, find the general solution of

$$(1-x)y'' + xy' - y = e^x(1-x)^2$$

$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = e^x(1-x)$

$W(e^x, x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1-x)$

$\int \frac{y_2 g}{W} = \int \frac{(x)(e^x(1-x))}{e^x(1-x)} dx = \int x\,dx = \frac{1}{2}x^2$

$\int \frac{y_1 g}{W} = \int \frac{(e^x)(e^x(1-x))}{e^x(1-x)} dx = \int e^x dx = e^x$

$y_p = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} = -(e^x)(\frac{1}{2}x^2) + (x)(e^x)$

$\boxed{y = C_1 e^x + C_2 x - \frac{1}{2}x^2 e^x + xe^x}$