

Exam 2

Math 217

This exam consists of 16 questions worth 5 points each. You must show all work. Answers without work will receive no credit. Recursion relations are acceptable for series solutions.

1. Solve $y^{(4)} + 2y''' - 2y'' + 2y' - 3y = 0$

$$r^4 + 2r^3 - 2r^2 + 2r - 3 = 0$$

$$\begin{array}{r|l} 1 & 1 & 2 & -2 & 2 & -3 \\ & & 1 & 3 & 1 & 3 \\ -3 & 1 & 3 & 1 & 3 & 0 \\ & & -3 & 0 & -3 & \\ \hline & 1 & 0 & 1 & & \end{array}$$

$$(r-1)(r+3)(r^2+1) = 0$$

$$y = c_1 e^t + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t$$

2. Solve $y''' + 2y'' - y' - 2y = 20 \sin(3t)$

$$r^3 + 2r^2 - r - 2 = 0$$

$$\begin{array}{r|l} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

$$(r-1)(r^2+3r+2) = 0$$

$$(r-1)(r+2)(r+1) = 0$$

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t} + y_p$$

$$y_p = A \cos(3t) + B \sin(3t)$$

$$y_p' = -3A \sin(3t) + 3B \cos(3t)$$

$$y_p'' = -9A \cos(3t) - 9B \sin(3t)$$

$$y_p''' = 27A \sin(3t) - 27B \cos(3t)$$

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t} + \frac{6}{13} \cos(3t) - \frac{4}{13} \sin(3t)$$

$$\begin{aligned} \mathcal{L}(y_p) &= (27A \sin(3t) - 27B \cos(3t)) + 2(-9A \cos(3t) - 9B \sin(3t)) \\ &\quad - (-3A \sin(3t) + 3B \cos(3t)) - 2(A \cos(3t) + B \sin(3t)) \\ &= 20 \sin 3t \end{aligned}$$

$$\begin{aligned} (27A - 18B + 3A - 2B) \sin(3t) + \\ (-27B - 18A - 3B - 2A) \cos(3t) &= 20 \sin 3t \end{aligned}$$

$$30A - 20B = 20$$

$$-20A - 30B = 0$$

$$3A - 2B = 2$$

$$2A + 3B = 0$$

$$A = -\frac{3}{2}B$$

$$3(-\frac{3}{2}B) - 2B = 2$$

$$-\frac{13}{2}B = 2$$

$$B = -\frac{4}{13}$$

$$A = \frac{6}{13}$$

$$y_p = \frac{6}{13} \cos 3t - \frac{4}{13} \sin 3t$$

3. Solve $y''' + 4y'' - 7y' - 10y = 0$

$$r^3 + 4r^2 - 7r - 10 = 0$$

$$2) \begin{array}{ccc|c} 1 & 4 & -7 & -10 \\ & 2 & 12 & 10 \\ \hline 1 & 6 & 5 & 0 \end{array}$$

$$(r-2)(r^2+6r+5)=0$$

$$(r-2)(r+5)(r+1)=0$$

$$y = c_1 e^{2t} + c_2 e^{-5t} + c_3 e^{-t}$$

4. Solve $y''' + y' = \tan t$

$$r^3 + r = 0$$

$$r(r^2 + 1) = 0$$

$$y_1 = 1 \quad y_2 = \cos t \quad y_3 = \sin t$$

$$W = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = \sin^2 t + \cos^2 t = 1$$

$$W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t$$

$$W_3 = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t$$

$$\int \frac{W_1 g}{W} = \int \tan t dt = \ln |\sec t|$$

$$\begin{aligned} \int \frac{W_2 g}{W} &= \int -\cos t \tan t dt \\ &= \int -\sin t dt = \cos t \end{aligned}$$

$$\begin{aligned} \int \frac{W_3 g}{W} &= \int -\sin t \tan t dt \\ &= \int -\frac{\sin^2 t}{\cos t} dt \\ &= \int \frac{\cos^2 t - 1}{\cos t} dt \\ &= \int \cos t - \sec t dt \\ &= \sin t - \ln |\sec t| + \tan t \end{aligned}$$

$$\begin{aligned} y_p &= y_1 \int \frac{W_1 g}{W} + y_2 \int \frac{W_2 g}{W} + y_3 \int \frac{W_3 g}{W} \\ &= \ln |\sec t| + \cos^2 t + \sin t (\sin t - \ln |\sec t| + \tan t) \\ &= \ln |\sec t| + 1 - \ln |\sec t| + \tan t \end{aligned}$$

$$y = c_1 + c_2 \cos t + c_3 \sin t + \ln |\sec t| + 1 - \ln |\sec t| + \tan t$$

5. A spring ($k=3$) has one end fixed to an immovable surface, and the other to the top of a 4 kg mass. Another spring ($k=2$) is attached to the bottom of the first mass, and has another 4 kg mass attached to its bottom end. Find an expression for the motion of the bottom mass as a function of time.



$$F_{S_1} = K_1 u_1 = 3u_1$$

$$F_{S_2} = K_2(u_1 - u_2) = 2(u_2 - u_1)$$

$$4u_1'' = -F_{S_1} + F_{S_2} = -3u_1 + 2(u_2 - u_1)$$

$$4u_2'' = -F_{S_2} = -2(u_2 - u_1)$$

$$4u_1'' + 5u_1 = 2u_2$$

$$4u_2'' + 2u_2 = 2u_1 \quad 2u_2'' + u_2 = u_1$$

$$4(2u_2'' + u_2)'' + 5(2u_2'' + u_2) = 2u_2$$

$$8u_2^{(4)} + 14u_2'' + 3u_2 = 0$$

$$8r^4 + 14r^2 + 3 = 0$$

$$(4r^2 + 1)(2r^2 + 3) = 0$$

$$r = \pm \frac{1}{2}i, \pm i\sqrt{\frac{3}{2}}$$

$$y = c_1 \cos \frac{t}{2} + c_2 \sin \frac{t}{2} + c_3 \cos(t\sqrt{\frac{3}{2}}) + c_4 \sin(t\sqrt{\frac{3}{2}})$$

6. Solve $2x^2y'' + 7xy' - 3y = 0$

$$x^2 y'' + \frac{7}{2} x y' - \frac{3}{2} y = 0$$

$$r^2 + \left(\frac{7}{2} - 1\right)r - \frac{3}{2} = 0$$

$$r^2 + \frac{5}{2}r - \frac{3}{2} = 0$$

$$2r^2 + 5r - 3 = 0$$

$$(2r - 1)(r + 3) = 0$$

$$r = \frac{1}{2}, r = -3$$

$$y = c_1 |x|^{1/2} + c_2 |x|^{-3}$$

7. Solve $x^2 y'' + y = 0$

$$r^2 + (0-1)r + 1 = 0$$

$$r^2 - r + 1 = 0$$

$$r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y = c_1 |x|^{1/2} \cos\left(\frac{\sqrt{3}}{2} \ln|x|\right) + c_2 |x|^{1/2} \sin\left(\frac{\sqrt{3}}{2} \ln|x|\right)$$

8. Solve $x^2 y'' - xy' + y = 0$

$$r^2 + (-1-1)r + 1 = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$y = c_1 |x| + c_2 |x| \ln|x|$$

9. List and classify the singularities of $x(x+3)^2(x-5)^2y'' + (x+3)y' + (x-5)y = 0$

$$\underline{x=0} : \lim_{x \rightarrow 0} x \cdot \frac{x+3}{x(x+3)^2(x-5)^2} = \frac{3}{3^2(-5)^2} \quad 0 \text{ is regular singular}$$

$$\lim_{x \rightarrow 0} x^2 \cdot \frac{(x-5)}{x(x+3)^2(x-5)^2} = \frac{-5}{3^2(-5)^2}$$

$$\underline{x=-3} : \lim_{x \rightarrow -3} (x+3) \cdot \frac{(x+3)}{x(x+3)^2(x-5)} = \frac{1}{-3(-8)} \quad -3 \text{ is regular singular}$$

$$\lim_{x \rightarrow -3} (x+3)^2 \cdot \frac{(x-5)}{x(x+3)^2(x-5)^2} = \frac{-8}{-3(-8)^2}$$

$$\underline{x=5} : \lim_{x \rightarrow 5} (x-5) \cdot \frac{(x+3)}{x(x+3)^2(x-5)^2} = \lim_{x \rightarrow 5} \frac{1}{x(x+3)(x-5)} = \text{DNE} \quad 5 \text{ is irregular singular}$$

10. Give a power series solution of $(x-1)y'' + 3xy' - y = 0$ centered at $x_0 = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$(x-1) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 3 \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$-2a_2 - a_0 = 0$$

$$(n+1)n a_{n+1} - (n+2)(n+1)a_{n+2} + 3n a_n - a_n = 0 \quad n \geq 1$$

$$a_{n+2} = \frac{n(n+1)a_{n+1} + (3n-1)a_n}{(n+2)(n+1)} \quad n \geq 1$$

$$a_2 = \frac{1}{2} a_0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

11. Give a power series solution of $xy'' + 3xy' - y = 0$ centered at $x_0 = 1$.

$$((x-1)+1) \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + (3(x-1)+3) \sum_{n=1}^{\infty} n a_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + 3 \sum_{n=1}^{\infty} n a_n(x-1)^n + 3 \sum_{n=1}^{\infty} n a_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)n a_{n+1}(x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + 3 \sum_{n=0}^{\infty} n a_n(x-1)^n + 3 \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} + 3n a_n + 3(n+1)a_{n+1} - a_n = 0$$

$$a_{n+2} = - \frac{(n+1)(n+3)a_{n+1} + (3n-1)a_n}{(n+2)(n+1)} \quad n \geq 0$$

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n$$

12. Give a power series solution of $xy'' - 3y' - y = 0, x > 0$, centered at $x_0 = 0$.

$$p_0 = \lim_{x \rightarrow 0} x \cdot \frac{-3}{x} = -3$$

$$q_0 = \lim_{x \rightarrow 0} x^2 \cdot \frac{-1}{x} = 0$$

$$r^2 - 4r = 0$$

$$r(r-4) = 0$$

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} - 3 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r+1)(n+r)a_{n+1} x^{n+r} - 3 \sum_{n=0}^{\infty} (n+r+1)a_{n+1} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$[(n+r+1) - 3]a_{n+1} x^{n+r} - a_n x^{n+r} = 0$$

$$y = x^4 \sum_{n=0}^{\infty} a_n x^n$$

$$a_{n+1} = \frac{a_n}{(n+5)(n+1)} \quad n \geq 0$$

13. Give a nontrivial lower bound for the radius of convergence of series solutions of

$$(x^3 - x^2 + x - 1)y'' + (x - 1)y' + y = 0$$

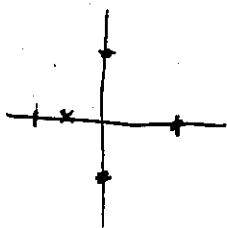
centered at $x_0 = -\frac{1}{2}$.

$$p = \frac{x-1}{(x-1)(x^2+1)} = \frac{1}{x^2+1}$$

poles at $\pm i$

$$q = \frac{1}{(x-1)(x^2+1)}$$

poles at $1, \pm i$



$$d(-\frac{1}{2}, i) = \sqrt{(-\frac{1}{2}-0)^2 + (0-1)^2} = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2}$$

$$d(-\frac{1}{2}, -i)$$

$$d(-\frac{1}{2}, 1) = \frac{3}{2} > \frac{\sqrt{5}}{2}$$

$$\boxed{R_{OC} \geq \frac{\sqrt{5}}{2}}$$

14. Give a nontrivial lower bound for the radius of convergence of series solutions of

$$(x^2 - 4)y'' + (x^2 + x - 2)y' + (x + 2)y = 0$$

centered at $x_0 = -1$.

$$p = \frac{(x+2)(x-1)}{(x+2)(x-2)} = \frac{x-1}{x-2}$$

pole at 2

$$q = \frac{x+2}{(x+2)(x-2)} = \frac{1}{x-2}$$

pole at 2

$$d(-1, 2) = 3$$

$$\boxed{R_{OC} \geq 3}$$

15. Show that $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

$$\begin{aligned}
 F(s) &= \mathcal{L}\{\cos 2t\} = \int_0^{\infty} e^{-st} \cos 2t \, dt \\
 &\quad u = \cos 2t \quad v = -\frac{1}{s} e^{-st} \\
 &\quad du = -2\sin 2t \, dt \quad dv = e^{-st} \, dt \\
 &= \lim_{a \rightarrow \infty} \left. -\frac{1}{s} e^{-st} \cos 2t \right|_0^a - \frac{2}{s} \int_0^{\infty} e^{-st} \sin 2t \, dt \\
 &= +\frac{1}{s} - \frac{2}{s} \int_0^{\infty} e^{-st} \sin 2t \, dt \\
 &\quad u = \sin 2t \quad v = -\frac{1}{s} e^{-st} \\
 &\quad du = 2\cos 2t \, dt \quad dv = e^{-st} \, dt \\
 &= +\frac{1}{s} - \frac{2}{s} \left(\lim_{a \rightarrow \infty} \left. -\frac{1}{s} e^{-st} \sin 2t \right|_0^a + \frac{2}{s} \int_0^{\infty} e^{-st} \cos 2t \, dt \right) \\
 &= +\frac{1}{s} - \frac{2}{s} \left(\frac{2}{s} F(s) \right) \\
 F(s) \left(1 + \frac{4}{s^2} \right) &= \frac{1}{s} \\
 F(s) &= \frac{s}{s^2+4}
 \end{aligned}$$

16. Show that $\mathcal{L}\{2t+t^2\} = \frac{2s+2}{s^3}$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} (2t+t^2) \, dt = 2 \int_0^{\infty} t e^{-st} \, dt + \int_0^{\infty} t^2 e^{-st} \, dt \\
 &\quad u = t \quad v = -\frac{1}{s} e^{-st} \\
 &\quad du = dt \quad dv = e^{-st} \, dt \\
 &= 2 \int_0^{\infty} t e^{-st} \, dt + \lim_{a \rightarrow \infty} \left. -\frac{1}{s} e^{-st} \cdot t^2 \right|_0^a + \frac{2}{s} \int_0^{\infty} t e^{-st} \, dt \\
 &= \left(2 + \frac{2}{s} \right) \int_0^{\infty} t e^{-st} \, dt \\
 &\quad u = t \quad v = -\frac{1}{s} e^{-st} \\
 &\quad du = dt \quad dv = e^{-st} \, dt \\
 &= \left(2 + \frac{2}{s} \right) \left(\lim_{a \rightarrow \infty} \left. -\frac{1}{s} e^{-st} \cdot t \right|_0^a + \frac{1}{s} \int_0^{\infty} e^{-st} \, dt \right) \\
 &= \frac{2s+2}{s^2} \int_0^{\infty} e^{-st} \, dt \\
 &= \frac{2s+2}{s^2} \lim_{a \rightarrow \infty} \left. -\frac{1}{s} e^{-st} \right|_0^{\infty} \\
 &= \frac{2s+2}{s^2} \cdot \frac{1}{s} = \frac{2s+2}{s^3}
 \end{aligned}$$