Exam 3

Math 217

This exam consists of 16 questions worth 5 points each. You must show all work. Answers without work will receive no credit. Do not leave answers in terms of convolutions.

1. Solve via Laplace transforms: $y'' + y = 8\cos 3t$, $y(0) = 0$, $y'(0) = 0$.

$s^2 Y + Y = 8 \cdot \frac{s}{s^2+9}$

$Y(s^2+1) = 8 \cdot \frac{s}{s^2+9}$

$Y = \frac{8s}{(s^2+1)(s^2+9)}$

$\frac{8s}{(s^2+1)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}$

$8s = (As+B)(s^2+9) + (Cs+D)(s^2+1)$

(1) $0 = A + C$ | (4) − (2) $8B = 0$, $B = 0$

(2) $0 = B + D$ | (2) $0 = 0 + D$

(3) $8 = 9A + C$ | (3) − (1) $8 = 8A$, $1 = A$

(4) $0 = 9B + D$ | (1) $0 = 1 + C$, $-1 = C$

$Y = \frac{s}{s^2+1} - \frac{s}{s^2+9}$

$\boxed{y = \cos t - \cos 3t}$

2. Solve via Laplace transforms: $y'' + 4y' + 3y = 18t$, $y(0) = -9$, $y'(0) = 9$

$(s^2Y - (-9)s - 9) + 4(sY - (-9)) + 3Y = \frac{18}{s^2}$

$(s^2 + 4s + 3)Y + 9s + 27 = \frac{18}{s^2}$

$Y = \frac{18}{s^2(s^2+4s+3)} - \frac{9(s+3)}{(s^2+4s+3)}$

$= \frac{18}{s^2(s+3)(s+1)} - \frac{9}{s+1}$

$\frac{18}{s^2(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{s+1}$

$18 = As(s+3)(s+1) + B(s+3)(s+1) + Cs^2(s+1) + Ds^2(s+3)$

$s = 0 \Rightarrow$ $18 = 3B$ $B = 6$

$s = -1 \Rightarrow$ $18 = 2D$ $D = 9$

$s = -3 \Rightarrow$ $18 = -18C$ $C = -1$

$s^3 \Rightarrow$ $0 = A + C + D$ $A = -8$

$Y = \left(\frac{-8}{s} + \frac{6}{s^2} - \frac{1}{s+3} + \frac{9}{s+1}\right) - \frac{9}{s+1}$

$= -\frac{8}{s} + \frac{6}{s^2} - \frac{1}{s+3}$

$\boxed{y = -8 + 6t - e^{-3t}}$

3. Compute $\mathcal{L}\{g(t)\}$ for $g(t) = \begin{cases} 0 & t \le 1 \\ t^3 & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$

$$g(t) = \left(u_1(t) - u_2(t)\right) \cdot t^3$$

$$= u_1(t)\left((t-1)+1\right)^3 - u_2(t)\left((t-2)+2\right)^3$$

$$= u_1(t)\left((t-1)^3 + 3(t-1)^2 + 3(t-1) + 1\right) - u_2(t)\left((t-2)^3 + 6(t-2)^2 + 12(t-2) + 8\right)$$

$$\boxed{\mathcal{L}\{g(t)\} = e^{-s}\left(\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}\right)}$$

4. Compute $\mathcal{L}^{-1}\{\frac{s^3+2s+5}{s^3-2s^2+5s}\}$

$$\frac{s^3+2s+5}{s^3-2s^2+5s} = 1 + \frac{2s^2-3s+5}{s^3-2s^2+5s}$$

$$\frac{2s^2-3s+5}{s^3-2s^2+5s} = \frac{A}{s} + \frac{Bs+C}{s^2-2s+5}$$

$$2s^2-3s+5 = A(s^2-2s+5) + (Bs+C)s$$

$$2 = A + B$$

$$-3 = -2A + C$$

$$5 = 5A$$

$A = 1$

$B = 1$

$C = -1$

$$\frac{s^3+2s+5}{s^3-2s^2+5s} = 1 + \frac{1}{s} + \frac{s-1}{(s-1)^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s^3+2s+5}{s^3-2s^2+5s}\right\} = \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$$

$$\boxed{= \delta(t) + 1 + e^{t}\cos 2t}$$

5. Show that $\sin 3t * \cos 3t = \frac{1}{2} t \sin 3t$.

$$\sin 3t * \cos 3t = \int_0^t \sin 3(t-\tau)\cos(3\tau)\,d\tau = \int_0^t \left(\sin(3t)\cos(3\tau) - \cos(3t)\sin(3\tau)\right)\cos(3\tau)\,d\tau$$

$$= \sin 3t \int_0^t \cos^2 3\tau \, d\tau - \cos 3t \int_0^t \sin 3\tau \cos 3\tau \, d\tau \qquad u = \sin 3\tau \quad du = 3\cos 3\tau\, d\tau$$

$$= \sin 3t \int_0^t \frac{1+\cos 6\tau}{2}\,d\tau - \cos 3t \left(\frac{1}{3} \cdot \int_0^{\sin 3t} u\,du \right)$$

$$= \sin 3t \left(\frac{1}{2}\tau + \frac{1}{2}\cdot\frac{1}{6}\sin 6\tau \right)\Big|_{\tau=0}^{\tau=t} - \frac{1}{6}\cos 3t \sin^2 3t$$

$$= \frac{1}{2} t \sin 3t + \frac{1}{2}\cdot\frac{1}{6}\sin 3t \sin 6t - \frac{1}{6}\cos 3t \sin^2 3t$$

$$= \frac{1}{2} t \sin 3t + \frac{1}{6}\left(\sin 3t (\sin 3t)(\cos 3t) - \cos 3t \sin^2 3t \right)$$

$$= \frac{1}{2} t \sin 3t$$

6. Solve via Laplace transforms: $y'' + 9y = 3\cos 3t$, $y(0) = 0$, $y'(0) = 0$

$$s^2 Y + 9 = 3 \frac{s}{s^2+9}$$

$$Y = \frac{3}{s^2+9} \cdot \frac{s}{s^2+9}$$

$$y = \sin 3t * \cos 3t$$

$$\boxed{y = \frac{1}{2} t \sin 3t}$$

7. Solve $y'' + y = u_3(t) - u_0(t)$, $y(0) = 0$, $y'(0) = 0$

$$s^2 Y + Y = \frac{e^{-3s}}{s} - \frac{1}{s}$$

$$Y = (e^{-3s} - 1) \cdot \frac{1}{s(s^2+1)}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs + C)s$$

$0 = A + B$

$0 = C$

$1 = A$

$$Y = (e^{-3s} - 1)\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$y = u_3(t)\big(1 - \cos(t-3)\big) - (1 - \cos t)$$

$$\boxed{= u_3(t)\big(1 - \cos(t-3)\big) + \cos t - 1}$$

8. Solve $2y'' + 2y' + \frac{17}{2}y = \delta(t-3)$, $y(0) = 1/2$, $y'(0) = -1/4$

$$2\left(s^2 Y - \tfrac{1}{2}s - \left(-\tfrac{1}{4}\right)\right) + 2\left(sY - \tfrac{1}{2}\right) + \tfrac{17}{2}Y = e^{-3s}$$

$$\left(2s^2 + 2s + \tfrac{17}{2}\right)Y - s - \tfrac{1}{2} = e^{-3s}$$

$$Y = e^{-3s} \cdot \frac{1}{2s^2+2s+\frac{17}{2}} + \frac{s+\frac{1}{2}}{2s^2+2s+\frac{17}{2}}$$

$$= \frac{1}{2}e^{-3s} \cdot \frac{1}{s^2+s+\frac{17}{4}} + \frac{1}{2} \cdot \frac{s+\frac{1}{2}}{s^2+s+\frac{17}{4}}$$

$$= \frac{1}{2}e^{-3s} \cdot \frac{1}{(s+\frac{1}{2})^2+4} + \frac{1}{2} \cdot \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+4}$$

$$= \frac{1}{4}e^{-3s} \cdot \frac{2}{(s+\frac{1}{2})^2+4} + \frac{1}{2} \cdot \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+4}$$

$$\boxed{y = \frac{1}{4}u_3(t)e^{-\frac{1}{2}(t-3)}\sin(2(t-3)) + \frac{1}{2}e^{-\frac{1}{2}t}\cos 2t}$$

9. Solve $\vec{x}' = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \vec{x}$

$$\lambda^2 - 3\lambda - 28 = 0$$

$$(\lambda - 7)(\lambda + 4) = 0$$

$$\lambda = 7, -4$$

$$(A - 7I)\vec{\xi} = 0 \qquad (A + 4I)\vec{\xi} = 0$$

$$\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \vec{\xi} = 0 \qquad \begin{pmatrix} 5 & 6 \\ 5 & 6 \end{pmatrix} \vec{\xi} = 0$$

$$\vec{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{\xi} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$$

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -6 \\ 5 \end{pmatrix} e^{-4t}}$$

10. Solve $\vec{x}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \vec{x}$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

$$(A - (1+2i)I)\vec{\xi} = 0$$

$$\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \vec{\xi} = 0$$

$$\vec{\xi} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$$

$$\boxed{\vec{x} = c_1 e^t \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right) + c_2 e^t \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t \right)}$$

11. Solve $\vec{x}' = \begin{pmatrix} -3 & -2 \\ 2 & -7 \end{pmatrix} \vec{x}$

$\lambda^2 + 10\lambda + 25 = 0$

$(\lambda + 5)^2 = 0$

$\lambda = -5$

$(A + 5I)\vec{\xi} = 0$ $\qquad$ $(A + 5I)\vec{\eta} = \vec{\xi}$

$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \vec{\xi} = 0$ $\qquad$ $\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\vec{\eta} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t} + c_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-5t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-5t} \right)}$$

12. Solve $\vec{x}' = \begin{pmatrix} 47 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$

$\lambda = 47, 2$

$(A - 47I)\vec{\xi} = 0$ $\qquad$ $(A - 2I)\vec{\xi} = 0$

$\begin{pmatrix} 0 & 0 \\ 0 & -45 \end{pmatrix} \vec{\xi} = 0$ $\qquad$ $\begin{pmatrix} 45 & 0 \\ 0 & 0 \end{pmatrix} \vec{\xi} = 0$

$\vec{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $\vec{\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{47t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}}$$

13. Solve $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$
$$\lambda = 2, -1$$

$(A - 2I)\vec{\varphi} = 0$ $\qquad$ $(A + I)\vec{\varphi} = 0$

$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\vec{\varphi} = 0$ $\qquad$ $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\vec{\varphi} = 0$

$\vec{\varphi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\vec{\varphi} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \vec{c}$$

$$\vec{c} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 6 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\boxed{\vec{x} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$$

14. Solve $\vec{x}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda - 4)(\lambda - 2) = 0$$
$$\lambda = 2, 4$$

$(A - 2I)\vec{\varphi} = 0$ $\qquad$ $(A - 4I)\vec{\varphi} = 0$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\vec{\varphi} = 0$ $\qquad$ $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\vec{\varphi} = 0$

$\vec{\varphi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad$ $\vec{\varphi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \vec{c}$$

$$\vec{c} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$$

15. Solve $\vec{x}' = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^t \\ 2t^2 \end{pmatrix}$.

$\lambda^2 - \frac{3}{2}\lambda - 1 = 0$

$2\lambda^2 - 3\lambda - 2 = 0$

$(2\lambda+1)(\lambda-2) = 0$

$\lambda = 2, \ -\frac{1}{2}$

$(A-2I)\vec{\varphi} = 0 \qquad (A+\frac{1}{2}I)\vec{\varphi} = 0$

$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ 1 & -1 \end{pmatrix}\vec{\varphi} = 0 \qquad \begin{pmatrix} 1 & \frac{3}{2} \\ 1 & \frac{3}{2} \end{pmatrix}\vec{\varphi} = 0$

$\vec{\varphi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{\varphi} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$

$\vec{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}e^{-\frac{1}{2}t} + \vec{x}_p$

$\vec{x}_p = \vec{a}e^t + \vec{b}t^2 + \vec{c}t + \vec{d}$

$\vec{a}e^t + 2\vec{b}t + \vec{c} = A\vec{a}e^t + A\vec{b}t^2 + A\vec{c}t + A\vec{d} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}e^t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}t^2$

$\vec{a} = A\vec{a} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$0 = A\vec{b} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$2\vec{b} = A\vec{c}$

$\vec{c} = A\vec{d}$

$\vec{a} = (A-I)^{-1}\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$= -\frac{2}{3}\begin{pmatrix} 0 & -\frac{3}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$= \left(-\frac{2}{3}\right)\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{4}{3} \end{pmatrix}$

$\vec{b} = A^{-1}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$= -\begin{pmatrix} 1 & -\frac{3}{2} \\ -1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$= -\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\vec{c} = 2A^{-1}\vec{b}$

$= -2\begin{pmatrix} 1 & -\frac{3}{2} \\ -1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$= -2\begin{pmatrix} -\frac{9}{2} \\ \frac{7}{2} \end{pmatrix} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$

$\vec{d} = A^{-1}\vec{c} = -\begin{pmatrix} 1 & -\frac{3}{2} \\ -1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 9 \\ -7 \end{pmatrix} = -\begin{pmatrix} \frac{39}{2} \\ -\frac{25}{2} \end{pmatrix} = \begin{pmatrix} -\frac{39}{2} \\ \frac{25}{2} \end{pmatrix}$

$$\boxed{\vec{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}e^{-\frac{1}{2}t} + \begin{pmatrix} 0 \\ -\frac{4}{3} \end{pmatrix}e^t + \begin{pmatrix} -3 \\ 1 \end{pmatrix}t^2 + \begin{pmatrix} 9 \\ -7 \end{pmatrix}t + \begin{pmatrix} -\frac{39}{2} \\ \frac{25}{2} \end{pmatrix}}$$

16. Solve $\vec{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t \sec t$.

$\lambda^2 - 1 = 0$

$(\lambda+1)(\lambda-1) = 0$

$\lambda = 1, -1$

$(A+I)\vec{\varphi} = 0 \qquad (A-I)\vec{\varphi} = 0$

$\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\vec{\varphi} = 0 \qquad \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\vec{\varphi} = 0$

$\vec{\varphi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{\varphi} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\vec{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^t$

$\Psi := \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}$

$\Psi^{-1} = \frac{1}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix}$

$\Psi^{-1}g = \frac{1}{2}\begin{pmatrix} e^{-t} & -e^{-t} \\ -e^t & 3e^t \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^t \sec t$

$= \frac{1}{2}\begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix}e^t \sec t$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sec t$

$\int \Psi^{-1}g\,dt = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\int \sec t\,dt$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ln|\sec t + \tan t|$

$\Psi\int\Psi^{-1}g\,dt = \begin{pmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\ln|\sec t + \tan t|$

$= \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}\ln|\sec t + \tan t|$

$$\boxed{\vec{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^t + \begin{pmatrix} 3e^t \\ e^t \end{pmatrix}\ln|\sec t + \tan t|}$$