Final Exam

Math 217

This exam consists of 16 questions worth 5 points each. You must show all work. Answers without work will receive no credit.

1. A 2kg mass is suspended from a spring, upon which the spring extends $\frac{10}{13}$ m. Friction in the spring exerts 8N of force when the mass is traveling 1 m/s. The mass is pushed up $\frac{1}{2}$m and released from rest. Give an expression for the motion of the mass as a function of time

$m\cdot g = K\cdot x = F_s$

$2\cdot 10 = K\cdot \frac{10}{13}$

$26 = K$

$F_f = bv$

$8 = b\cdot 1$

$8 = b$

$x(0) = -\frac{1}{2} \quad x'(0) = 0$

$-\frac{1}{2} = x(0) = c_1$

$0 = x'(0) = -2c_1 + 3c_2$

$= 1 + 3c_2$

$-\frac{1}{3} = c_2$

$2x'' + 8x' + 26x = 0$

$x'' + 4x' + 13x = 0$

$r^2 + 4r + 13 = 0$

$r = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$

$x = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$

$$\boxed{x = -\frac{1}{2}e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t}$$

2. A 200L tank contains a 5 g/L salt solution. A 20 g/L salt solution is pumped in at 10 L/min and the (well mixed) tank is drained at 10 L/min. Find an expression for the amount of salt in the tank as a function of time.

$\frac{dQ}{dt}$ = rate in − rate out

$= (20)(10) - \left(\frac{Q}{200}\right)(10)$

$= 200 - \frac{Q}{20}$

$Q(0) = (200)(5) = 1000$

$20Q' = 4000 - Q$

$\frac{Q'}{4000-Q} = \frac{1}{20}$

$-\ln|4000-Q| = \frac{1}{20}t + c$

$4000 - Q = Ae^{-\frac{1}{20}t}$

$Q = 4000 - Ae^{-\frac{1}{20}t}$

$1000 = Q(0) = 4000 - A$

$A = 3000$

$$\boxed{Q = 4000 - 3000e^{-\frac{1}{20}t}}$$

3. Solve $\frac{dy}{dx} = 1 - \frac{y}{2x}$, $y(1) = \frac{11}{3}$.

$$y' + \frac{1}{2x} \cdot y = 1$$

$$\mu = e^{\int \frac{1}{2x}} = e^{\frac{1}{2}\ln|x|} = x^{1/2}$$

$$x^{1/2} y' + \frac{1}{2} x^{-1/2} y = x^{1/2}$$

$$(x^{1/2} y)' = x^{1/2}$$

$$x^{1/2} y = \frac{2}{3} x^{\frac{3}{2}} + c$$

$$y = \frac{2}{3} x + c x^{-1/2}$$

$$\frac{11}{3} = y(1) = \frac{2}{3} + c$$

$$3 = c$$

$$\boxed{y = \frac{2}{3} x + 3x^{-1/2}}$$

4. Given that $y_1 = e^x$ and $y_2 = x$ are solutions of the corresponding homogeneous equation, find the general solution of

$$(1-x)y'' + xy' - y = e^x(1-x)^2$$

$$y'' + \frac{x}{1-x} y' - \frac{1}{1-x} y = e^x(1-x)$$

$$W(y_1, y_2) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x = e^x(1-x)$$

$$y_p = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W} = -e^x \int \frac{x\,[e^x(1-x)]}{e^x(1-x)}\,dx + x \int \frac{e^x\,[e^x(1-x)]}{e^x(1-x)}\,dx$$

$$= -e^x \int x\,dx + x \int e^x\,dx$$

$$= -e^x \cdot \frac{1}{2}x^2 + xe^x$$

$$y_p = xe^x\left(1 - \frac{1}{2}x\right)$$

$$\boxed{y = C_1 e^x + C_2 x + xe^x\left(1 - \frac{1}{2}x\right)}$$

5. Given that $y_1 = e^x$ is a solution, find the general solution of $(1-x)y'' + xy' - y = 0$.

$y = v(x)\cdot e^x$

$y' = (v' + v)e^x$

$y'' = (v'' + 2v' + v)e^x$

$(1-x)(v'' + 2v' + v)e^x + x(v' + v)e^x - ve^x = 0$

$(1-x)e^x v'' + [(1-x)\cdot 2e^x + xe^x]v' + [(1-x)e^x + xe^x - e^x]v = 0$

$(1-x)e^x v'' + (2-x)e^x v' = 0$

$\frac{v''}{v'} = -\frac{x-2}{x-1}$

$\ln|v'| = -\int \frac{x-2}{x-1}\,dx = -\int 1 - \frac{1}{x-1}\,dx$

$= -x + \ln|x-1| + A$

$v' = C_1 (x+1)e^{-x}$

$v = \int C_1 (x+1) e^{-x} = C_1 x e^{-x} + C_2$

$y = v\cdot e^x = \boxed{C_1\cdot x + C_2\cdot e^x}$

6. Solve $y^{(3)} - 3y' + 2y = 10\sin t$

$r^3 - 3r + 2 = 0$

$$\begin{array}{r|rrr|r} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline 1 & 1 & 1 & -2 & 0 \\ & & 1 & 2 & \\ \hline -2 & 1 & 2 & 0 & \\ & & -2 & & \\ \hline & 1 & 0 & & \end{array}$$

$(r-1)^2(r+2) = 0$

$y = C_1 e^t + C_2 t e^t + C_3 e^{-2t} + y_p$

$y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$y_p''' = A\sin t - B\cos t$

$(A\sin t - B\cos t) - 3(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) = 10\sin t$

$(4A + 2B)\sin t + (-4B + 2A)\cos t = 10\sin t$

$4A + 2B = 10$

$2A - 4B = 0$

$A = 2B$

$4(2B) + 2B = 10$

$B = 1$

$A = 2$

$\boxed{y = C_1 e^t + C_2 t e^t + C_3 e^{-2t} + 2\cos t + \sin t}$

7. Solve $x^2y'' + 3xy' + y = 0$, $x > 0$, $y(1) = 2$, $y'(1) = 3$

$r^2 + (3-1)r + 1 = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$y = c_1 x^{-1} + c_2 x^{-1} \ln x$

$2 = y(1) = c_1$

$y = 2x^{-1} + c_2 x^{-1} \ln x$

$y' = -2x^{-2} + c_2(-x^{-2}\ln x + x^{-2})$

$3 = y'(1) = -2 + c_2$

$5 = c_2$

$$\boxed{y = \frac{2}{x} + \frac{5\ln x}{x}}$$

8. Give a power series solution of $xy'' + 3xy' - y = 0$ centered at $x_0 = 1$

1 is an ordinary point $\qquad y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$$x\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + 3x\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$[(x-1)+1]\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + [3(x-1)+3]\sum_{n=1}^{\infty} n a_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + 3\sum_{n=1}^{\infty} n a_n(x-1)^n + 3\sum_{n=1}^{\infty} n a_n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=1}^{\infty} (n+1)n a_{n+1}(x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + 3\sum_{n=1}^{\infty} n a_n(x-1)^n + 3\sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)n a_{n+1}(x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + 3\sum_{n=0}^{\infty} n a_n(x-1)^n + 3\sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} + 3n a_n + 3(n+1)a_{n+1} - a_n\right](x-1)^n = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)(n+3)a_{n+1} + (3n-1)a_n = 0$$

$$\boxed{y = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad a_{n+2} = -\frac{n+3}{n+2}a_{n+1} - \frac{3n-1}{(n+2)(n+1)}a_n}$$

9. Give (and justify) a nontrivial lower bound for the radius of convergence of series solutions of

$$(x^3 + x^2 + x + 1)y'' + (x^2 + 1)y' + y = 0$$

centered at $x_0 = 1$.

$(x+1)(x^2+1)y'' + (x^2+1)y' + y = 0$

$\frac{x^2+1}{(x+1)(x^2+1)} = \frac{1}{x+1} \Rightarrow -1$

$\frac{1}{(x+1)(x^2+1)} = \frac{1}{(x+1)(x^2+1)} \Rightarrow -1, \pm i$

$d(1, -1) = 2$

$d(1, \pm i) = \sqrt{2}$

$\boxed{\text{ROC} \geq \sqrt{2}}$

10. Show that $\mathcal{L}\{e^{5t}\} = \frac{1}{s-5}$.

$\mathcal{L}\{e^{5t}\} = \int_0^\infty e^{-st} \cdot e^{5t}\, dt = \int_0^\infty e^{t(5-s)}\, dt$

$= \lim_{a\to\infty} \frac{1}{5-s} e^{t(5-s)} \Big|_{t=0}^{t=a} = \lim_{a\to\infty} \frac{1}{5-s} e^{a(5-s)} - \frac{1}{5-s}$

$= \frac{1}{s-5}$

11. Show $\sin 5t * \cos 5t = \frac{1}{2}t\sin 5t$

$$\sin 5t * \cos 5t = \int_0^t \sin 5(t-\tau)\cos 5\tau \, d\tau = \int_0^t \sin(5t-5\tau)\cos 5\tau \, d\tau$$

$$= \int_0^t (\sin 5t \cos 5\tau - \cos 5t \sin 5\tau)\cos 5\tau \, d\tau$$

$$= \sin 5t \int_0^t \cos^2 5\tau \, d\tau - \cos 5t \int_0^t \sin 5\tau \cos 5\tau \, d\tau \qquad \begin{array}{l} u = \sin 5\tau \\ du = 5\cos 5\tau \, d\tau \end{array}$$

$$= \sin 5t \int_0^t \frac{1}{2}(\cos(2\cdot 5\tau)+1)\,d\tau - \cos 5t \int_0^{\sin 5t} u \cdot \frac{du}{5}$$

$$= \sin 5t \cdot \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{5}\sin(2\cdot 5\tau) + \tau\right)\Big|_{\tau=0}^{\tau=t} - \frac{1}{5}\cos 5t \cdot \frac{1}{2}u^2\Big|_{u=0}^{u=\sin 5t}$$

$$= \frac{1}{2}\sin 5t\left(\frac{1}{5}\sin 5t\cos 5t + t\right) - \frac{1}{5}\cos 5t \cdot \frac{1}{2}\sin^2 5t$$

$$= \frac{1}{2}t\sin 5t$$

12. Solve $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 0$ for $g(t) = \begin{cases} 0 & t \le 1 \\ t-1 & 1 \le t \le 3 \\ 2 & t \ge 3 \end{cases}$

$$g(t) = [u_1(t) - u_3(t)](t-1) + 2u_3(t)$$

$$= u_1(t)(t-1) + u_3(t)(2-(t-1))$$

$$= u_1(t)(t-1) - u_3(t)(t-3)$$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$s=0 \Rightarrow \quad 1 = B$

$s^3 \Rightarrow \quad A + C = 0 \quad C=0$

$s^2 \Rightarrow \quad B + D = 0 \quad D = -1$

$s \Rightarrow \quad A = 0$

$$\mathcal{L}\{y''+y\} = \mathcal{L}(g)$$

$$s^2Y + Y = e^{-s}\cdot\frac{1}{s^2} - e^{-3s}\cdot\frac{1}{s^2}$$

$$Y = (e^{-s} - e^{-3s})\cdot\frac{1}{s^2(s^2+1)}$$

$$= (e^{-s} - e^{-3s})\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$$

$$y = u_1(t)\big((t-1) - \sin(t-1)\big) - u_3(t)\big((t-3) - \sin(t-3)\big)$$

13. Solve $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$, $y(0) = 0$, $y'(0) = 0$.

$$s^2 Y + 4Y = e^{-\pi s} - e^{-2\pi s}$$

$$Y = \left(e^{-\pi s} - e^{-2\pi s}\right) \cdot \frac{1}{s^2+4} = \frac{1}{2}\left(e^{-\pi s} - e^{-2\pi s}\right) \cdot \frac{2}{s^2+4}$$

$$y = \frac{1}{2} U_\pi(t) \sin(2(t-\pi)) - \frac{1}{2} U_{2\pi}(t) \sin(2(t-2\pi))$$

$$= \frac{1}{2} U_\pi(t) \sin 2t - \frac{1}{2} U_{2\pi}(t) \sin 2t$$

$$= \frac{1}{2} \sin 2t \left(U_\pi(t) - U_{2\pi}(t)\right)$$

14. Solve $\vec{x}' = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-3)(\lambda-2) = 0$$

$$\lambda = 2, 3$$

$(A-2I)\vec{\varphi} = 0$ $\qquad$ $(A-3I)\vec{\varphi} = 0$

$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}\vec{\varphi} = 0$ $\qquad$ $\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix}\vec{\varphi} = 0$

$\vec{\varphi} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\qquad$ $\vec{\varphi} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= -\begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$$

$c_1 = -6 \quad c_2 = 8$

$$\vec{x} = -6 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + 8 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t}$$

$$\boxed{= \begin{pmatrix} 12 \\ -6 \end{pmatrix} e^{2t} + \begin{pmatrix} -8 \\ 8 \end{pmatrix} e^{3t}}$$

15. List and classify the critical points of $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x(1-x-y) \\ y(\frac{3}{4} - y - \frac{1}{2}x) \end{pmatrix}$.

$x(1-x-y)=0$

$y(\frac{3}{4}-y-\frac{1}{2}x)=0$

$x=0 \;:\quad y(\frac{3}{4}-y)=0$

$y=0 \quad y=\frac{3}{4}$

$1-x-y=0 \;:\quad y(\frac{3}{4}-y-\frac{1}{2}(1-y))=0$

$y(\frac{1}{4}-\frac{1}{2}y)=0$

$y=0 \qquad y=\frac{1}{2}$

$x=1 \qquad x=\frac{1}{2}$

$$J = \begin{pmatrix} 1-2x-y & -x \\ -\frac{1}{2}y & \frac{3}{4}-2y-\frac{1}{2}x \end{pmatrix}$$

$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \quad \lambda = 1, \frac{3}{4}$

$J(0,\frac{3}{4}) = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{pmatrix} \quad \lambda = \frac{1}{4}, -\frac{3}{4}$

$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{pmatrix} \quad \lambda = -1, \frac{1}{4}$

$J(\frac{1}{2},\frac{1}{2}) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}$

$\lambda^2 + \lambda + \frac{1}{8} = 0$

$\lambda = \frac{-1 \pm \sqrt{1-\frac{1}{2}}}{2} = \frac{-1 \pm \sqrt{\frac{1}{2}}}{2} < 0$

$(0,0)$	unstable node
$(0,\frac{3}{4})$	saddle point
$(1,0)$	saddle point
$(\frac{1}{2},\frac{1}{2})$	asymptotically stable node

16. Solve $y'' + y = 0$, $y(0) = 1$, $y(\frac{\pi}{2}) = 2$.

$r^2+1=0$

$r = \pm i$

$y = c_1 \cos t + c_2 \sin t$

$1 = c_1 \cdot 1 + c_2 \cdot 0 = c_1$

$2 = c_1 \cdot 0 + c_2 \cdot 1 = c_2$

$\boxed{y = \cos t + 2\sin t}$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28