Problem Set 1, Solutions:

1.(i) $z = \sqrt{3} + i$ If $z = re^{i\theta}$ in polar form:

$$r = \sqrt{3+1} = 2 \qquad \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

so $z = 2e^{i\frac{\pi}{6}}$, so

$$z^8 = 2^8 e^{i\frac{4\pi}{3}} = 256\, e^{i\frac{4\pi}{3}}$$
$$= 256\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= -128 - i\, 128\sqrt{3}$$

(ii) $1 - i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$

$$(1-i)^{-16} = \sqrt{2}^{-16}\, e^{i\frac{\pi}{4}\times 16} = 2^{-8} e^{i4\pi}$$
$$= \frac{1}{2^8}$$

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(2) (i) $z = re^{i\theta}$

$$z^4 = -25 = 25e^{i\pi}$$

so $r^4 = 25 \qquad 4\theta = \pi + 2k\pi \qquad k = 0, 1, 2, 3$

so $r = \sqrt{5} \qquad \theta = \frac{\pi}{4} + \frac{k\pi}{2} \qquad k = 0, 1, 2, 3$

The 4 roots are:

$$\sqrt{5}\, e^{i\frac{\pi}{4}} = \sqrt{5}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{\sqrt{10}}{2} + i\frac{\sqrt{10}}{2}$$

$$\sqrt{5}\, e^{i\frac{\pi}{4} + i\frac{\pi}{2}} = -\frac{\sqrt{10}}{2} + \frac{i\sqrt{10}}{2}$$

$$\sqrt{5}\, e^{i\frac{\pi}{4} + i\pi} = -\frac{\sqrt{10}}{2} - \frac{i\sqrt{10}}{2}$$

$$\sqrt{5}\, e^{i\frac{\pi}{4} + \frac{3\pi}{2}} \quad \sqrt{10} \quad i\sqrt{10}$$

(ii) $(\sqrt{3} - i)^{\frac{1}{3}}$:

If $z = re^{i\theta}$ and $z^3 = \sqrt{3} - i = 2e^{-\frac{i\pi}{6}}$

so $r^3 = 2$ and $3\theta = -\frac{i\pi}{6} + 2K\pi$ $\quad K = 0, 1, 2$

so the 3 roots are:

$\sqrt[3]{2}\, e^{-\frac{i\pi}{18}} = \sqrt[3]{2} \cos\frac{\pi}{18} + i\sqrt[3]{2} \sin\frac{-\pi}{18}$

$\sqrt[3]{2}\, e^{-\frac{i\pi}{18} + i\frac{2\pi}{3}} = \sqrt[3]{2} \cos\frac{11\pi}{18} + i\sqrt[3]{2} \sin\frac{11\pi}{18}$

$\sqrt[3]{2}\, e^{+i(-\frac{\pi}{18} + \frac{4\pi}{3})} = \sqrt[3]{2} \cos\frac{23\pi}{18} + i\sqrt[3]{2} \sin\frac{23\pi}{18}$

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3. (i) $|z - (2-i)| \le 2$: This is a circle of radius 2 around $2-i$

(ii) $z = x + iy$, $\bar{z} = x - iy$. $2z - 2\bar{z} = 2x + 2iy - 2x + 2iy$
$= 4iy$

$4iy = 3$ is not possible since y is a real number,
so $2z - 2\bar{z} = 3$ does not have any solution

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4. Write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

To show that $|z_1 + z_2| \le |z_1| + |z_2|$, we show

(*) $|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$. We have:

$|z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

$(|z_1| + |z_2|)^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$$= (x_1^2+y_1^2) + (x_2^2+y_2^2) + 2\sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2}$$

So to show (*), we need to show

$$(x_1+x_2)^2+(y_1+y_2)^2 \le (x_1^2+y_1^2)+(x_2^2+y_2^2)+2\sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_1^2},$$

or equivalently (after cancelling terms).

$$(**) \qquad \cancel{2}x_1x_2+\cancel{2}y_1y_2 \le \cancel{2}\sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2}$$

we have

$$(x_1x_2+y_1y_2)^2 = x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2$$

$$\left(\sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2}\right)^2 = x_1^2x_2^2+x_1^2y_2^2+x_2^2y_1^2+y_1^2y_2^2$$

and since $x_2^2y_1^2+x_1^2y_2^2-2x_1x_2y_1y_2 = (x_2y_1-x_1y_2)^2 \ge 0$,
we get

$$(x_1x_2+y_1y_2)^2 \le \left(\sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2}\right)^2$$

and so

$$x_1x_2+y_1y_2 \le |x_1x_2+y_1y_2| \le \sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2}.$$

So (**) and hence (*) follow.

5. $i = e^{\frac{i\pi}{2}}$

$-2-2i = \sqrt{8}\, e^{i\frac{5\pi}{4}}$

So $\dfrac{i}{-2-2i} = \dfrac{1}{\sqrt{8}}\, e^{i\left(\frac{\pi}{2}-\frac{5\pi}{4}\right)} = \dfrac{1}{\sqrt{8}}\, e^{-i\frac{3\pi}{4}}$ $\quad -\pi < -\frac{3\pi}{4} \leq \pi$

Therefore, the principal value of the argument is $\boxed{-\frac{3\pi}{4}}$

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6. To say $|z-z_0| = k$ is to say $|z-z_0|^2 = k^2$

For any complex number z, $|z|^2 = z\bar{z}$.

Therefore $|z-z_0|^2 = k^2$ exactly when $(z-z_0)(\overline{z-z_0}) = k^2$,

that is exactly when $(z-z_0)(\bar{z}-\bar{z}_0) = k^2$.

But $(z-z_0)(\bar{z}-\bar{z}_0) = \underbrace{z\bar{z}}_{=} \underbrace{- z\bar{z}_0 - z_0\bar{z}}_{=} + \underbrace{z_0\bar{z}_0}_{=}$

$= |z|^2 - 2\operatorname{Re} z z_0 + |z_0|^2$