Math 416
Solutions to problem set # 11
Fall 2009

As for the points $z = \pm n\pi$ $(n = 1, 2, \ldots, N)$, write

$$\frac{1}{z^2 \sin z} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = z^2 \sin z.$$

Since

$$p(\pm n\pi) = 1 \neq 0, \quad q(\pm n\pi) = 0, \quad \text{and} \quad q'(\pm n\pi) = n^2\pi^2 \cos n\pi = (-1)^n n^2 \pi^2 \neq 0,$$

it follows that

$$\operatorname*{Res}_{z=\pm n\pi} \frac{1}{z^2 \sin z} = \frac{1}{(-1)^n n^2 \pi^2} \cdot \frac{(-1)^n}{(-1)^n} = \frac{(-1)^n}{n^2 \pi^2}.$$

So, by the residue theorem,

$$\int_{C_N} \frac{dz}{z^2 \sin z} dz = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right].$$

Rewriting this equation in the form

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i} \int_{C_N} \frac{dz}{z^2 \sin z}$$

and recalling from Exercise 7, Sec. 41, that the value of the integral here tends to zero as N tends to infinity, we arrive at the desired summation formula:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Question 1

~~6.~~ The path C here is the positively oriented boundary of the rectangle with vertices at the points ± 2 and $\pm 2 + i$. The problem is to evaluate the integral

$$\int_C \frac{dz}{(z^2 - 1)^2 + 3}.$$

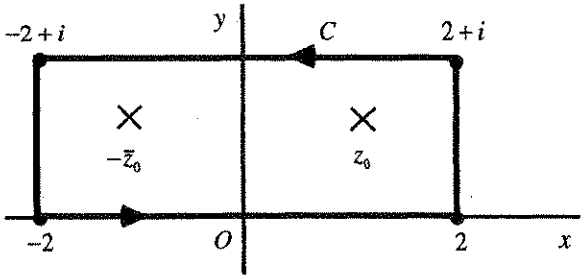
The isolated singularities of the integrand are the zeros of the polynomial

$$q(z) = (z^2 - 1)^2 + 3.$$

Setting this polynomial equal to zero and solving for z^2, we find that any zero z of $q(z)$ has the property $z^2 = 1 \pm \sqrt{3}i$. It is straightforward to find the two square roots of $1 + \sqrt{3}i$ and also the two square roots of $1 - \sqrt{3}i$. These are the four zeros of $q(z)$. Only two of those zeros,

$$z_0 = \sqrt{2}e^{i\pi/6} = \frac{\sqrt{3} + i}{\sqrt{2}} \quad \text{and} \quad -\overline{z}_0 = -\sqrt{2}e^{-i\pi/6} = \frac{-\sqrt{3} + i}{\sqrt{2}},$$

lie inside C. They are shown in the figure below.



To find the residues at z_0 and $-\overline{z}_0$, we write the integrand of the integral to be evaluated as

$$\frac{1}{(z^2 - 1)^2 + 3} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = (z^2 - 1)^2 + 3.$$

This polynomial $q(z)$ is, of course, the same $q(z)$ as above; hence $q(z_0) = 0$. Note, too, that p and q are analytic at z_0 and that $p(z_0) \neq 0$. Finally, it is straightforward to show that $q'(z) = 4z(z^2 - 1)$ and hence that

$$q'(z_0) = 4z_0(z_0^2 - 1) = -2\sqrt{6} + 6\sqrt{2}i \neq 0.$$

We may conclude, then, that z_0 is a simple pole of the integrand, with residue

$$\frac{p(z_0)}{q'(z_0)} = \frac{1}{-2\sqrt{6} + 6\sqrt{2}i}.$$

Similar results are to be found at the singular point $-\overline{z}_0$. To be specific, it is easy to see that

$$q'(-\overline{z}_0) = -q'(\overline{z}_0) = -\overline{q'(z_0)} = 2\sqrt{6} + 6\sqrt{2}i \neq 0,$$

the residue of the integrand at $-\overline{z}_0$ being

$$\frac{p(-\overline{z}_0)}{q'(-\overline{z}_0)} = \frac{1}{2\sqrt{6} + 6\sqrt{2}i}.$$

Finally, by the residue theorem,

$$\int_C \frac{dz}{(z^2-1)^2+3} = 2\pi i\left(\frac{1}{-2\sqrt{6}+6\sqrt{2}i} + \frac{1}{2\sqrt{6}+6\sqrt{2}i}\right) = \frac{\pi}{2\sqrt{2}}.$$

7. We are given that $f(z) = 1/[q(z)]^2$, where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. These conditions on q tell us that q has a zero of order $m = 1$ at z_0. Hence $q(z) = (z - z_0)g(z)$, where g is a function that is analytic and nonzero at z_0; and this enables us to write

$$f(z) = \frac{\phi(z)}{(z-z_0)^2}, \quad \text{where} \quad \phi(z) = \frac{1}{[g(z)]^2}.$$

So f has a pole of order 2 at z_0, and

$$\operatorname*{Res}_{z=z_0} f(z) = \phi'(z_0) = -\frac{2g'(z_0)}{[g(z_0)]^3}.$$

But, since $q(z) = (z - z_0)g(z)$, we know that

$$q'(z) = (z-z_0)g'(z) + g(z) \quad \text{and} \quad q''(z) = (z-z_0)g''(z) + 2g'(z).$$

Then, by setting $z = z_0$ in these last two equations, we find that

$$q'(z_0) = g(z_0) \quad \text{and} \quad q''(z_0) = 2g'(z_0).$$

Consequently, our expression for the residue of f at z_0 can be put in the desired form:

$$\operatorname*{Res}_{z=0} f(z) = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

8. *(a)* To find the residue of the function $\csc^2 z$ at $z = 0$, we write

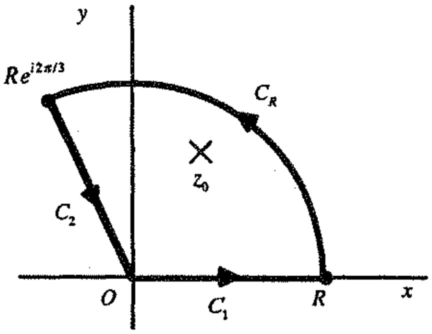
$$\csc^2 z = \frac{1}{[q(z)]^2}, \quad \text{where} \quad q(z) = \sin z.$$

Since q is entire, $q(0) = 0$, and $q'(0) = 1 \neq 0$, the result in Exercise 7 tells us that

$$\operatorname*{Res}_{z=0} \csc^2 z = -\frac{q''(0)}{[q'(0)]^3} = 0.$$

Question 2

~~8.~~ The problem here is to establish the integration formula $\int\limits_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$ using the simple closed contour shown below, where $R > 1$.



There is only one singularity of the function $f(z) = \frac{1}{z^3+1}$, namely $z_0 = e^{i\pi/3}$, that is interior to the closed contour when $R > 1$. According to the residue theorem,

$$\int_{C_1} \frac{dz}{z^3+1} + \int_{C_R} \frac{dz}{z^3+1} + \int_{C_2} \frac{dz}{z^3+1} = 2\pi i \operatorname*{Res}_{z=z_0} \frac{1}{z^3+1},$$

where the legs of the closed contour are as indicated in the figure. Since C_1 has parametric representation $z = r$ $(0 \le r \le R)$,

$$\int_{C_1} \frac{dz}{z^3+1} = \int\limits_0^R \frac{dr}{r^3+1};$$

and, since $-C_2$ can be represented by $z = re^{i2\pi/3}$ $(0 \le r \le R)$,

$$\int_{C_2} \frac{dz}{z^3+1} = -\int_{-C_2} \frac{dz}{z^3+1} = -\int\limits_0^R \frac{e^{i2\pi/3}\,dr}{(re^{i2\pi/3})^3+1} = -e^{i2\pi/3}\int\limits_0^R \frac{dr}{r^3+1}.$$

Furthermore,

$$\operatorname*{Res}_{z=z_0} \frac{1}{z^3+1} = \frac{1}{3z_0^2} = \frac{1}{3e^{i2\pi/3}}.$$

Consequently,

$$(1-e^{i2\pi/3})\int\limits_0^R \frac{dr}{r^3+1} = \frac{2\pi i}{3e^{i2\pi/3}} - \int_{C_R} \frac{dz}{z^3+1}.$$

But

$$\left|\int_{C_R} \frac{dz}{z^3+1}\right| \le \frac{1}{R^3-1}\cdot\frac{2\pi R}{3} \to 0 \text{ as } R \to \infty.$$

This gives us the desired result, with the variable of integration r instead of x:

$$\int\limits_0^R \frac{dr}{r^3+1} = \frac{2\pi i}{3(e^{i2\pi/3} - e^{i4\pi/3}\cdot e^{-i6\pi/3})} = \frac{2\pi i}{3(e^{i2\pi/3} - e^{-i2\pi/3})} = \frac{\pi}{3\sin(2\pi/3)} = \frac{2\pi}{3\sqrt{3}}.$$

According to the residue theorem,

$$\int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)} + \int_{C_R} f(z)e^{iz}dz = 2\pi i(B_1+B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=ai}[f(z)e^{iz}] = \left.\frac{e^{iz}}{(z+ai)(z^2+b^2)}\right]_{z=ai} = \frac{e^{-a}}{2a(b^2-a^2)i}$$

and

$$B_2 = \operatorname*{Res}_{z=bi}[f(z)e^{iz}] = \left.\frac{e^{iz}}{(z^2+a^2)(z+bi)}\right]_{z=bi} = \frac{e^{-b}}{2b(a^2-b^2)i}.$$

That is,

$$\int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) - \int_{C_R} f(z)e^{iz}dz,$$

or

$$\int_{-R}^{R} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) - \operatorname{Re}\int_{C_R} f(z)e^{iz}dz.$$

Now, if z is a point on C_R,

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{1}{(R^2-a^2)(R^2-b^2)}$$

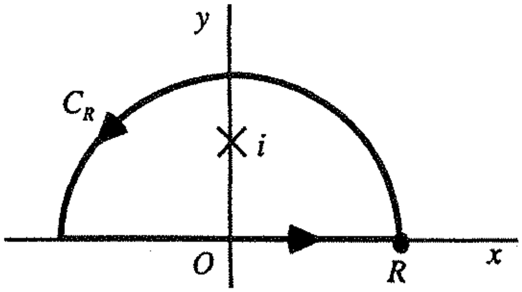
and $|e^{iz}| = e^{-y} \le 1$. Hence

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iz}\,dz\right| \le \left|\int_{C_R} f(z)e^{iz}\,dz\right| \le M_R\pi R = \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \to 0 \text{ as } R\to\infty.$$

So it follows that

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) \qquad (a>b>0).$$

Question 3. ~~2.~~ This problem is to evaluate the integral $\int_0^{\infty} \frac{\cos ax}{x^2+1}dx$, where $a \ge 0$. The function $f(z) = \frac{1}{z^2+1}$ has the singularities $\pm i$; and so we may integrate around the simple closed contour shown below, where $R > 1$.



We start with

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}dx + \int_{C_R} f(z)e^{iaz}dz = 2\pi i B,$$

where

$$B = \operatorname*{Res}_{z=i}\left[f(z)e^{iaz}\right] = \left.\frac{e^{iaz}}{z+i}\right]_{z=i} = \frac{e^{-a}}{2i}.$$

Hence

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}dx = \pi e^{-a} - \int_{C_R} f(z)e^{iaz}dz,$$

or

$$\int_{-R}^{R} \frac{\cos ax}{x^2+1}dx = \pi e^{-a} - \operatorname{Re}\int_{C_R} f(z)e^{iaz}dz,$$

Since

$$|f(z)| \le M_R \text{ where } \quad M_R = \frac{1}{R^2-1},$$

we know that

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iaz}dz\right| \le \left|\int_{C_R} f(z)e^{iaz}dz\right| \le \frac{\pi R}{R^2-1};$$

and so

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx = \pi e^{-a}.$$

That is,

$$\int_{0}^{\infty} \frac{\cos ax}{x^2+1}dx = \frac{\pi}{2}e^{-a} \qquad (a \ge 0).$$

4. To evaluate the integral $\int_0^{\infty} \frac{x\sin 2x}{x^2+3}dx$, we first introduce the function

$$f(z) = \frac{z}{z^2+3} = \frac{z}{(z-z_1)(z-\bar{z}_1)},$$

where $z_1 = \sqrt{3}i$. The point z_1 lies above the x axis, and $\bar{z}_1$ lies below it. If we write

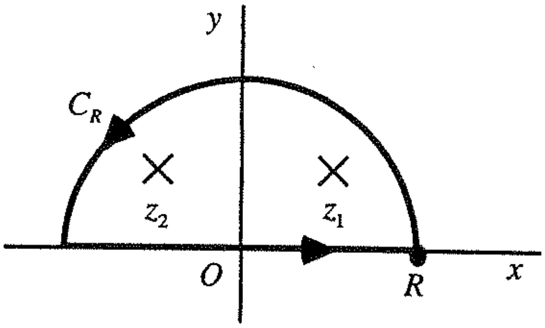
$$f(z)e^{i2z} = \frac{\phi(z)}{z-z_1} \quad \text{where} \quad \phi(z) = \frac{z\exp(i2z)}{z-\bar{z}_1},$$

Question 4.

~~6.~~ The integral to be evaluated is $\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx$, where $a > 0$. We define the function $f(z) = \frac{z^3}{z^4+4}$; and, by computing the fourth roots of -4, we find that the singularities

$$z_1 = \sqrt{2}e^{i\pi/4} = 1+i \quad \text{and} \quad z_2 = \sqrt{2}e^{i3\pi/4} = \sqrt{2}e^{i\pi/4}e^{i\pi/2} = (1+i)i = -1+i$$

both lie inside the simple closed contour shown below, where $R > \sqrt{2}$. The other two singularities lie below the real axis.



The residue theorem and the method of Theorem 2 in Sec. 69 for finding residues at simple poles tell us that

$$\int_{-R}^{R} \frac{x^3 e^{iax}}{x^4+4}\,dx + \int_{C_R} f(z)e^{iaz}\,dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_1^3 e^{iaz_1}}{4z_1^3} = \frac{e^{iaz_1}}{4} = \frac{e^{ia(1+i)}}{4} = \frac{e^{-a}e^{ia}}{4}$$

and

$$B_2 = \operatorname*{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_2^3 e^{iaz_2}}{4z_2^3} = \frac{e^{iaz_2}}{4} = \frac{e^{ia(-1+i)}}{4} = \frac{e^{-a}e^{-ia}}{4}.$$

Since

$$2\pi i(B_1 + B_2) = \pi i e^{-a}\left(\frac{e^{ia}+e^{-ia}}{2}\right) = i\pi e^{-a}\cos a,$$

we are now able to write

$$\int_{-R}^{R} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a - \operatorname{Im}\int_{C_R} f(z)e^{iaz}\,dz.$$

Furthermore, if z is a point on C_R, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R^3}{R^4 - 4} \to 0 \text{ as } R \to \infty;$$

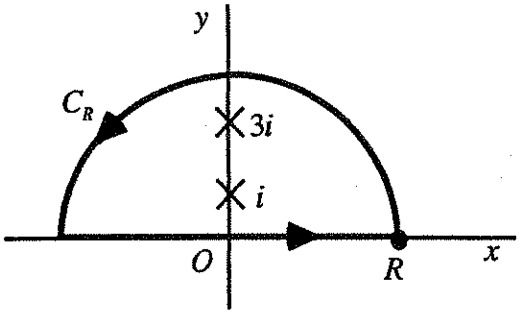
and this means that

$$\left|\operatorname{Im} \int_{C_R} f(z)e^{iaz}dz\right| \le \left|\int_{C_R} f(z)e^{iaz}dz\right| \to 0 \text{ as } R \to \infty,$$

according to limit (1), Sec. 74. Finally, then,

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx = \pi e^{-a} \cos a \qquad (a > 0).$$

8. In order to evaluate the integral $\int_0^\infty \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)}$, we introduce here the function $f(z) = \frac{z^3}{(z^2+1)(z^2+9)}$. Its singularities in the upper half plane are i and $3i$, and we consider the simple closed contour shown below, where $R > 3$.



Since

$$\operatorname*{Res}_{z=i}\left[f(z)e^{iz}\right] = \left.\frac{z^3 e^{iz}}{(z+i)(z^2+9)}\right]_{z=i} = -\frac{1}{16e}$$

and

$$\operatorname*{Res}_{z=3i}\left[f(z)e^{iz}\right] = \left.\frac{z^3 e^{iz}}{(z^2+1)(z+3i)}\right]_{z=3i} = \frac{9}{16e^3},$$

the residue theorem tells us that

$$\int_{-R}^{R} \frac{x^3 e^{ix}\, dx}{(x^2+1)(x^2+9)} + \int_{C_R} f(z)e^{iz}dx = 2\pi i\left(-\frac{1}{16e} + \frac{9}{16e^3}\right),$$

or

$$\int_{-R}^{R} \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8e}\left(\frac{9}{e^2} - 1\right) - \operatorname{Im} \int_{C_R} f(z)e^{iz}dz.$$

Limit (1), Sec. 74, then tells us that

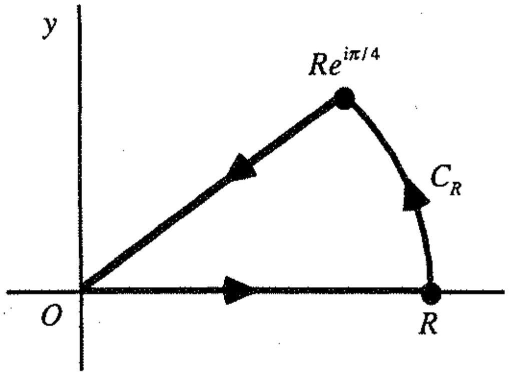
$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \to 0 \text{ as } R \to \infty,$$

and so

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2).$$

Question 5

~~12.~~ *(a)* Since the function $f(z) = \exp(iz^2)$ is entire, the Cauchy-Goursat theorem tells us that its integral around the positively oriented boundary of the sector $0 \le r \le R,\ 0 \le \theta \le \pi/4$ has value zero. The closed path is shown below.



A parametric representation of the horizontal line segment from the origin to the point R is $z = x\ (0 \le x \le R)$, and a representation for the segment from the origin to the point $Re^{i\pi/4}$ is $z = re^{i\pi/4}\ (0 \le r \le R)$. Thus

$$\int_0^R e^{ix^2}dx + \int_{C_R} e^{iz^2}dz - e^{i\pi/4}\int_0^R e^{-r^2}dr = 0,$$

or

$$\int_0^R e^{ix^2}dx = e^{i\pi/4}\int_0^R e^{-r^2}dr - \int_{C_R} e^{iz^2}dz.$$

By equating real parts and then imaginary parts on each side of this last equation, we see that

$$\int_0^R \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr - \operatorname{Re}\int_{C_R} e^{iz^2}dz$$

and

$$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr - \operatorname{Im}\int_{C_R} e^{iz^2}dz.$$

(b) A parametric representation for the arc C_R is $z = Re^{i\theta}$ $(0 \le \theta \le \pi/4)$. Hence

$$\int_{C_R} e^{iz^2}\,dz = \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} Rie^{i\theta}\,d\theta = iR\int_0^{\pi/4} e^{-R^2\sin 2\theta} e^{iR^2\cos 2\theta} e^{i\theta}\,d\theta.$$

Since $\left|e^{iR^2\cos 2\theta}\right| = 1$ and $\left|e^{i\theta}\right| = 1$, it follows that

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta.$$

Then, by making the substitution $\phi = 2\theta$ in this last integral and referring to the form (3), Sec. 74, of Jordan's inequality, we find that

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi \le \frac{R}{2}\cdot\frac{\pi}{2R^2} = \frac{\pi}{4R} \to 0 \text{ as } R \to \infty.$$

(c) In view of the result in part *(b)* and the integration formula

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

it follows from the last two equations in part *(a)* that

$$\int_0^{\infty} \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^{\infty} \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

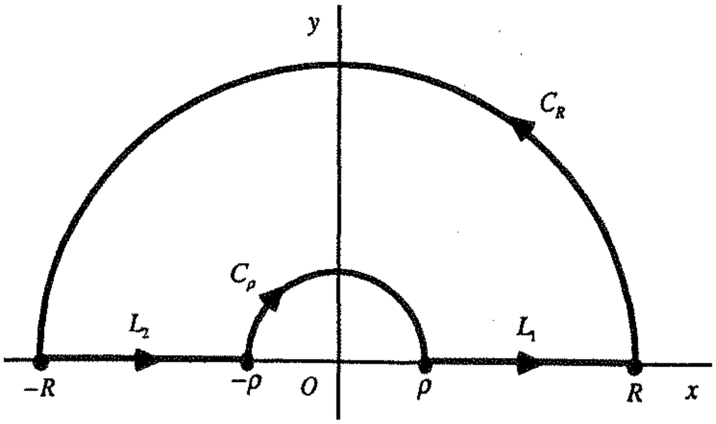
SECTION 77

Question 6.

. ~~1~~. The main problem here is to derive the integration formula

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0, b \ge 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 0,$$

or

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since L_1 and $-L_2$ have parametric representations

$$L_1 : z = re^{i0} = r\ (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r\ (\rho \le r \le R),$$

we can see that

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_\rho^R \frac{e^{iar} - e^{ibr}}{r^2}\,dr + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{r^2}\,dr$$

$$= \int_\rho^R \frac{(e^{iar} + e^{-iar}) - (e^{ibr} + e^{-ibr})}{r^2}\,dr = 2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}\,dr.$$

Thus

$$2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}\,dr = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

In order to find the limit of the first integral on the right here as $\rho \to 0$, we write

$$f(z) = \frac{1}{z^2}\left[\left(1 + \frac{iaz}{1!} + \frac{(iaz)^2}{2!} + \frac{(iaz)^3}{3!} + \cdots\right) - \left(1 + \frac{ibz}{1!} + \frac{(ibz)^2}{2!} + \frac{(ibz)^3}{3!} + \cdots\right)\right]$$

$$= \frac{i(a-b)}{z} + \cdots \quad (0 < |z| < \infty).$$

From this we see that $z = 0$ is a simple pole of $f(z)$, with residue $B_0 = i(a-b)$. Thus

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = -B_0 \pi i = -i(a-b)\pi i = \pi(a-b).$$

As for the limit of the value of the second integral as $R \to \infty$, we note that if z is a point on C_R, then

$$f(z) \le \frac{|e^{iaz}|+|e^{ibz}|}{|z|^2} = \frac{e^{-ay}+e^{-by}}{R^2} \le \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.$$

It is now clear that letting $\rho \to 0$ and $R \to \infty$ yields

$$2\int_0^\infty \frac{\cos(ar)-\cos(br)}{r^2}\,dr = \pi(b-a).$$

This is the desired integration formula, with the variable of integration r instead of x. Observe that when $a = 0$ and $b = 2$, that result becomes

$$\int_0^\infty \frac{1-\cos(2x)}{x^2}\,dx = \pi.$$

But $\cos(2x) = 1 - 2\sin^2 x$, and we arrive at

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

2. Let us derive the integration formula

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{(1-a)\pi}{4\cos(a\pi/2)} \qquad (-1 < a < 3),$$

where $x^a = \exp(a \ln x)$ when $x > 0$. We shall integrate the function

$$f(z) = \frac{z^a}{(z^2+1)^2} = \frac{\exp(a \log z)}{(z^2+1)^2} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right),$$

whose branch cut is the origin and the negative imaginary axis, around the simple closed path shown below.