1.

(i) $f(z) = 4z^3 + z^2 - 5z + 1$,

$z = x + iy$

So $f(z) = 4(x+iy)^3 + (x+iy)^2 - 5(x+iy) + 1$

$= 4x^3 + 4i^3y^3 + 12x^2iy + 12xi^2y^2 + x^2 + i^2y^2 + 2xiy - 5x - 5iy$

$= (4x^3 - 12xy^2 + x^2 - y^2 - 5x + 1) + i(-4y^3 + 12x^2y + 2xy - 5y)$

(ii) $f(z) = \dfrac{1}{z^2 - 1} = \dfrac{1}{(x+iy)^2 - 1} = \dfrac{1}{(x^2 - y^2 - 1) + i(2xy)}$

multiplying the numerator and the denominator by $(x^2 - y^2 - 1) - i(2xy)$, we get:

$$f(z) = \frac{(x^2-y^2-1) - i(2xy)}{(x^2-y^2-1)^2 + (2xy)^2} = \frac{x^2-y^2-1}{(x^2-y^2-1)^2 + 4x^2y^2} - i\,\frac{2xy}{(x^2-y^2-1)^2 + 4x^2y^2}$$

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2. Replacing x with $\frac{z+\bar{z}}{2}$ and y with $\frac{z-\bar{z}}{2i}$, we get

$f(z) = x^2 - y^2 - 2y + i(2x - 2xy) = \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 - 2\left(\frac{z-\bar{z}}{2i}\right) + i\left(z + \bar{z} - \frac{(z+\bar{z})(z}{2i}\right.$

$= \dfrac{z^2 + \bar{z}^2 + 2z\bar{z} + z^2 + \bar{z}^2 - 2z\bar{z}}{4} + i\left(z - \bar{z} + z + \bar{z} - \dfrac{z^2 - \bar{z}^2}{2i}\right) = \boxed{\bar{z}^2 + 2iz}$

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3. (i) You can either argue that since $\lim_{z \to i} \frac{z^2+1}{3i} = \frac{i^2+1}{3i} = 0$,

$\lim_{z \to i} \frac{3i}{z^2+1} = \infty$, or use the δ, M argument as follows:

we want to show for every $M > 0$, there is a δ such that whenever $|z - i| < \delta$, $\left|\frac{3i}{z^2+1}\right| > M$.

Note that

$$\left|\frac{3i}{z^2+1}\right| = \frac{|3i|}{|z-i||z+i|} = \frac{3}{|z-i|\,|z+i|}.$$

Also

$$|z+i| \underset{\uparrow}{\leq} |z-i| + |i-(-i)| = |z-i| + 2.$$

triangular inequality

so if $|z-i|<1$, $|z+i| < 1+2 = 3$, so $\frac{3}{|z+i|} > \frac{3}{3} = 1$

Therefore, if we choose $\delta = \min(\frac{1}{M}, 1)$, we get

If $|z-i| < \delta$, then

$$\left|\frac{3i}{z^2+1}\right| = \frac{1}{|z-i|} \cdot \left|\frac{3}{z+i}\right| > \frac{1}{\frac{1}{M}} \cdot 1 = M.$$

(ii) Again, you can either argue that. since if $f(z) = \frac{3z^2}{(z-1)^2}$,

$$f(\tfrac{1}{z}) = \frac{3 \cdot \frac{1}{z^2}}{(\frac{1}{z} - 1)^2} = \frac{\frac{3}{z^2}}{\frac{(1-z)^2}{z^2}} = \frac{3}{(1-z)^2}, \text{ and since.}$$

$\lim_{z\to 0} f(\frac{1}{z}) = \lim_{z\to 0} \frac{3}{(1-z)^2} = 3$, we can conclude that $\lim_{z\to\infty} f(z) = 3$.

You can also use the M, ε argument as follows:

we want to show for every $\varepsilon > 0$, there is $M > 0$ such that when $|z| > M$, $\left|\frac{3z^2}{(z-1)^2} - 3\right| < \varepsilon$

$$\left|\frac{3z^2}{(z-1)^2} - 3\right| = \left|\frac{3(2z-1)}{(z-1)^2}\right| = \left|\frac{3}{z-1}\right| \left|\frac{2z-1}{z-1}\right|$$

Note that $|z| \underset{\uparrow}{\leq} |z-1| + 1$, so $|z-1| \geq |z| - 1$

triangular inequality

so if $|z| > N+1$, $N \geq 1$, we get $|z-1| > N$ and

and $|2z-1| \le |2z-2|+1 \le 2|z-1|+1 < 3|z-1|$ (if $|z|>2$, so $|z-1|>1$)

therefore $\left|\frac{2z-1}{z-1}\right| < 3$

so if we choose $M > Max\left(\frac{9}{\varepsilon}+1, 2\right)$ for a given ε, we get

$|z-1| > |z|-1 > \frac{9}{\varepsilon}$, and $|z-1|>1$, so $\left|\frac{2z-1}{z-1}\right| < 3$

so $\left|\frac{3}{z-1}\right|\left|\frac{2z-1}{z-1}\right| < \frac{9}{|z-1|} < \frac{9}{\frac{9}{\varepsilon}} = \varepsilon.$

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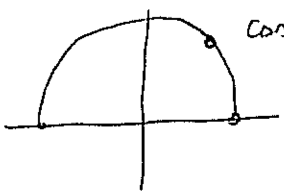
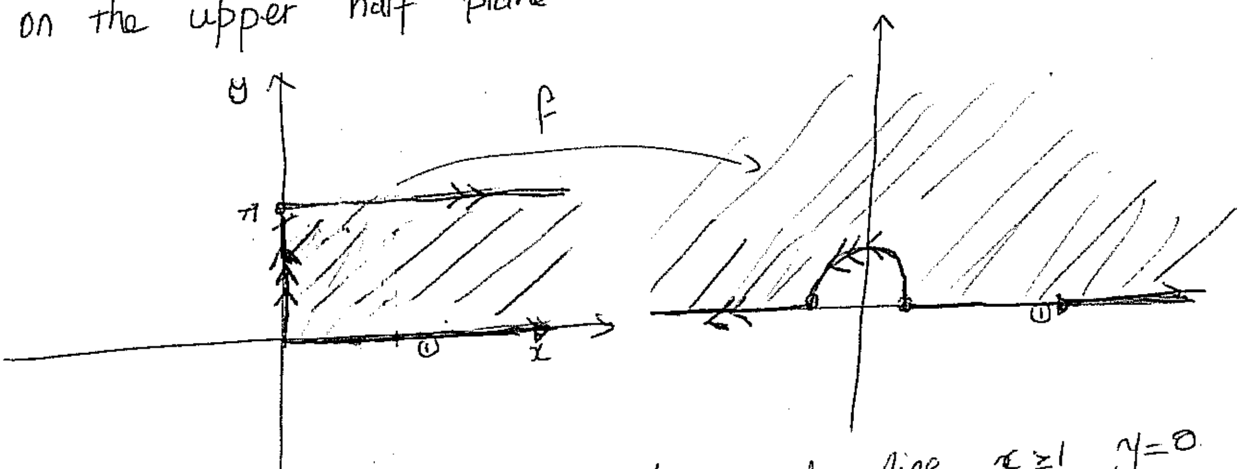
4. $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$

Since $x \ge 0$, $e^x \ge 1$, and since $0 \le y \le \pi$, $\cos y + i\sin y$ is a point on the upper half othe unit circle.



so $e^x(\cos y + i\sin y)$ can be any point outside the unit circle on the upper half plane



The line $y=0$ $x\ge 0$ is mapped to the line $x \ge 1$ $y=0$
The line $y=\pi$ $x \ge 0$ is mapped to the line $x \le -1$ $y=0$
and the segment $x=0$ $0 \le y \le \pi$ is mapped to the upper half circle

5. we want to show for every $\varepsilon > 0$ there is $\delta > 0$ such that when $|z - z_0| < \delta$ $|\mathrm{Re}\, z - \mathrm{Re}\, z_0| < $.

Note that $|z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

$$|\mathrm{Re}\, z - \mathrm{Re}\, z_0| = |x - x_0| \leq \sqrt{(x-x_0)^2 + (y-y_0)^2} = |z - z_0|$$

if we $\delta = \varepsilon$

when $|z - z_0| < \delta = \varepsilon$, $|\mathrm{Re}\, z - \mathrm{Re}\, z_0| < |z - z_0| < \varepsilon$

(ii) We want to show for every $\varepsilon > 0$, there is $\delta > 0$ such that when $|z| < \delta$, $\left|\frac{5z^3}{\bar{z}}\right| < \varepsilon$. Note that

$$\left|\frac{5z^3}{\bar{z}}\right| = \frac{5|z|^3}{|\bar{z}|} = 5|z|^2$$

so if $\delta = \sqrt{\frac{\varepsilon}{5}}$, we get

if $|z| < \delta$ then $|z|^2 < \delta^2 = \frac{\varepsilon}{5}$, so $|5z|^2 < \varepsilon$.

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6. $z_0 = x_0 + iy_0$ $z = x + iy$

$$f'(z_0) = \lim_{z \to z_0} \frac{\mathrm{Im}\, z - \mathrm{Im}\, z_0}{z - z_0} = \lim_{z \to z_0} \frac{y - y_0}{(x - x_0) + i(y - y_0)}$$

If z approaches z_0 along the horizontal line, so $y = y_0$, we get

$$\lim_{z \to z_0} \frac{y_0 - y_0}{(x - x_0) + i(y_0 - y_0)} = \lim_{z \to z_0} \frac{1}{x - x_0} = \infty$$

If z approaches z_0 along the vertical line, so $x = x_0$, we get

$$\lim_{z \to z_0} \frac{y - y_0}{(x_0 - x_0) + i(y - y_0)} = \frac{1}{i} = -i,$$

therefore, the derivative does not exist